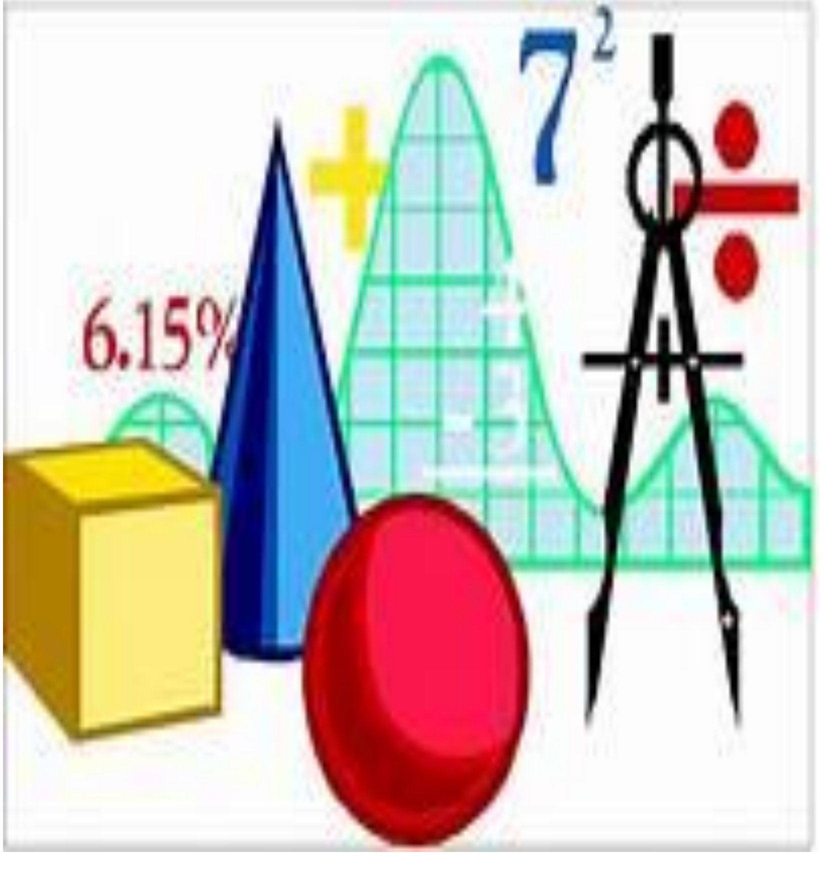
University of Anbar
College of Engineering
Mechanical Engineering Dept.

ME 2202 – Calculus IV

2021-2022 Second Semester



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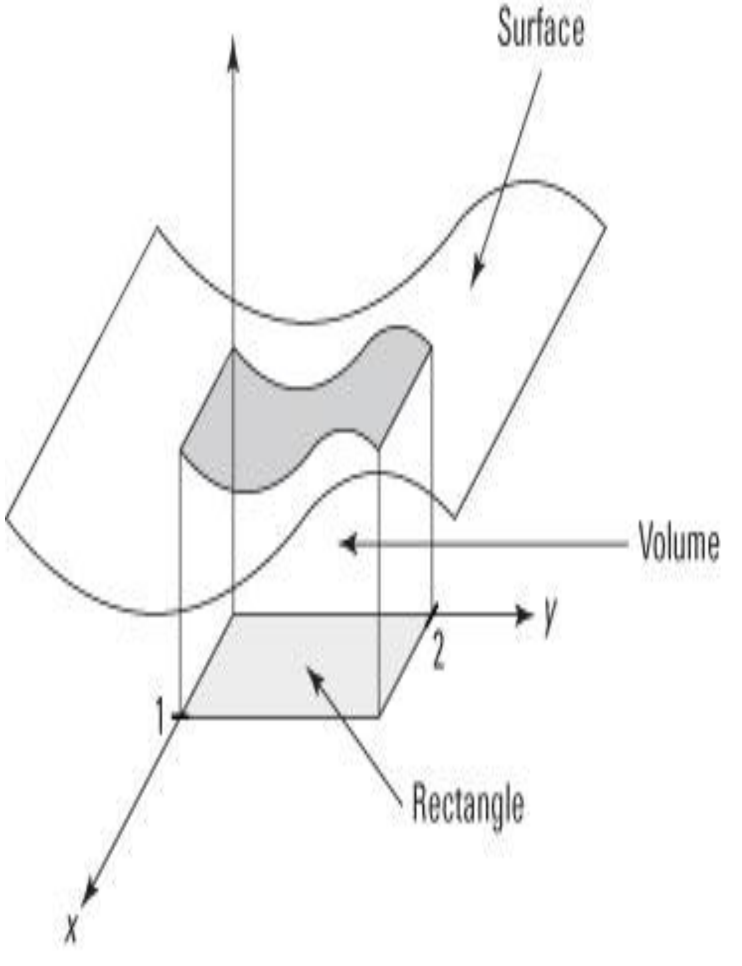
Chapter One

Multiple Integrals



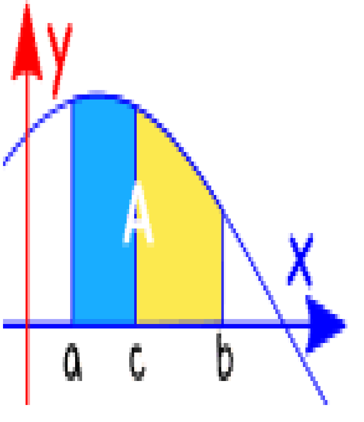
1.1 introduction

In this chapter the idea of a definite integral is extend to multiple integrals (double & triple) of functions of two or three variable. These idea are then used to compute volume, mass, etc.

1.2 Double Integrals

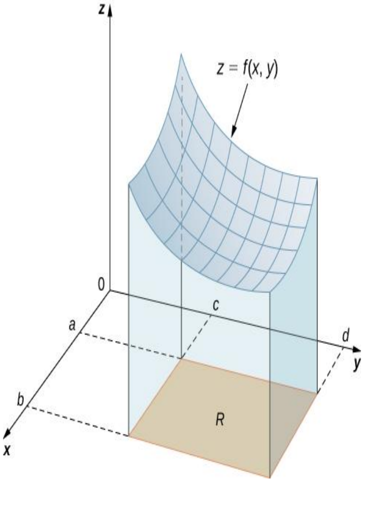
In general, single integral when there is one variable and it can be described as

$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{\infty} (f_{xi})\Delta x$$



In a similar manner we consider a function f of two variables defined on a closed rectangle.

$$\int\int_R f(x,y)dA = \lim_{m,n\to\infty} \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} f(xij,yij)\,\Delta A$$



1.2.1 Properties of Double Integrals

$$1-\iint_R [f(x,y) \mp g(x,y)]\, dA = \iint_R f(x,y)dA \mp \iint_R g(x,y)dA$$

$$2-\iint_R C\, f(x,y)dA = C\iint_R f(x,y)dA \quad \text{where C is a constant}$$

$$3-if\ R = R_1 + R_2$$

$$\iint_R f(x,y)dA = \iint_{R1} f(x,y)dA + \iint_{R2} f(x,y)dA$$

$$4-Area = A = \iint_R dA \quad ,dA = dxdy = dydx$$

$$5-volume = S = \iint_R f(x,y)dA \quad ,where f(x,y) = z$$

Example 1.1

Evaluate

$$(a)\int_{0}^{3}\int_{1}^{2} x^2y\,dydx \qquad (b)\int_{1}^{2}\int_{0}^{3} x^2y\,dxdy$$

Solution

(a)

$$\int_{0}^{3}\int_{1}^{2} x^2y\,dydx = \int_0^3\left[\int_1^2 x^2y\,dy\right]dx = \int_0^3\left[\frac{x^2y^2}{2}\right]_1^2 dx$$

(b)

$$= \int_0^3 \frac{3}{2}x^2dx = \frac{x^3}{3}\Bigg]_0^3 = \frac{27}{2}$$

$$\int_{1}^{2}\int_{0}^{3} x^2y\,dxdy = \int_1^2\left[\int_0^3 x^2y\,dx\right]dy = \int_1^2\left[\frac{x^3y}{3}\right]_0^3 dy$$

$$= \int_1^2 9\,ydy = 9\frac{y^2}{2}\Bigg]_1^2 = \frac{27}{2}$$

Example 1.2

Evaluate $\iint_R y\sin(xy)\,dA \quad where\ R = [1,2] \times [0,\pi]$

Solution

$$\iint y\sin(xy)\,dA = \int_0^{\pi}\int_1^{2} y\sin(xy)\,dx\,dy = \int_0^{\pi}[-\cos(xy)]_1^2\,dy$$

H.W.

$$\int_1^2\int_0^{\pi} y\sin(xy)\,dy\,dx$$

$$= \int_0^{\pi}(-\cos 2y + \cos y)\,dy = -\frac{1}{2}\sin 2y + \sin y]_0^{\pi} = 0$$

1.3 Double integrals over general regions

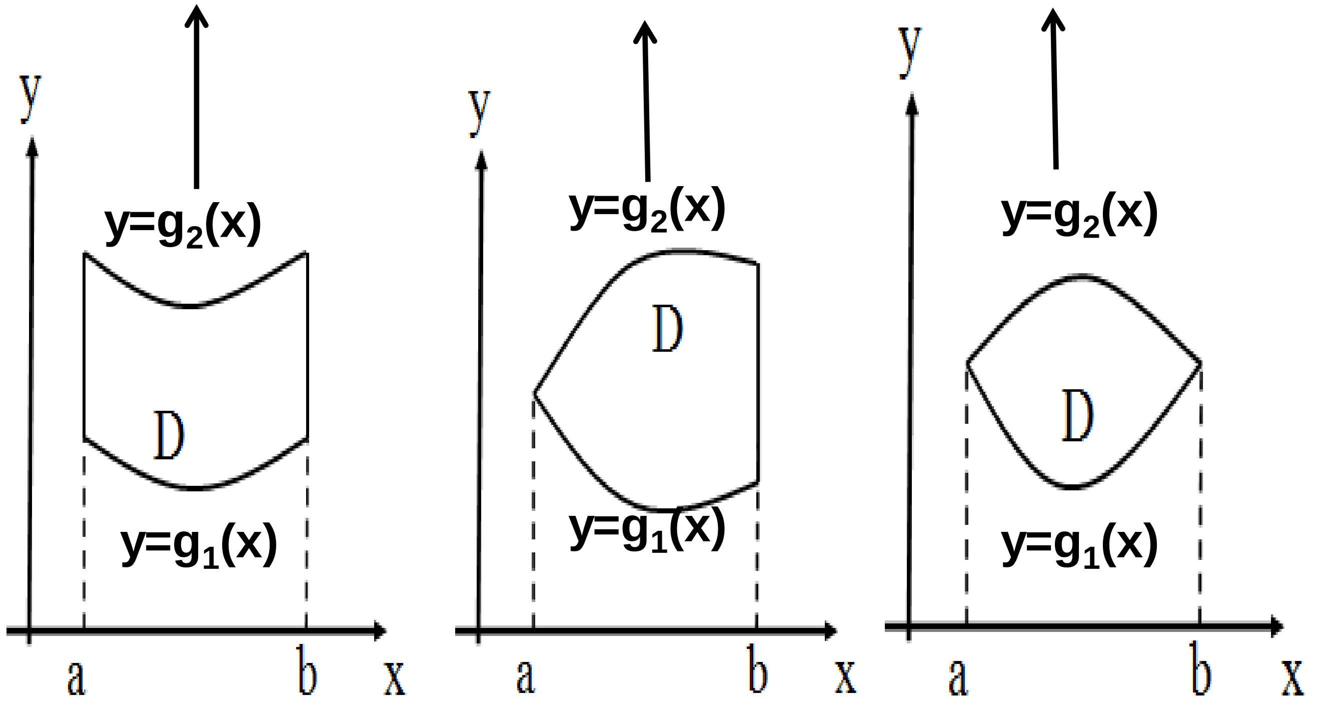
In double integrals, it can be integrate a function f not just a rectangles but also over region D of more general shape.

1.3.1 Type one

The plane region D is said to be type one if it lies between the graph of two continuous functions of x that is

$$D = \{(x,y) a \leq x \leq b, g1(x) \leq y \leq g2(x)\}$$

$$\iint_D f(x,y)\,dA = \int_a^b \int_{g1(x)}^{g2(x)} f(x,y)\,dy\,dx$$

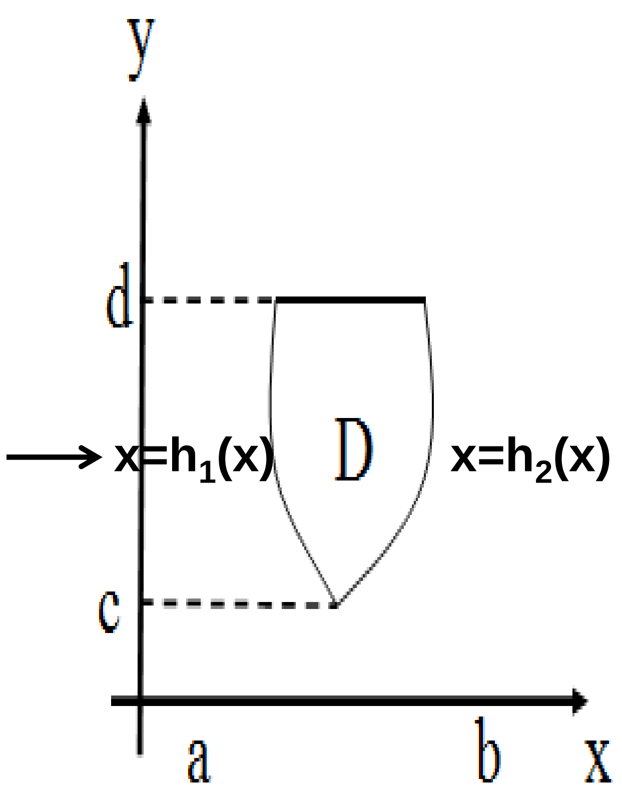


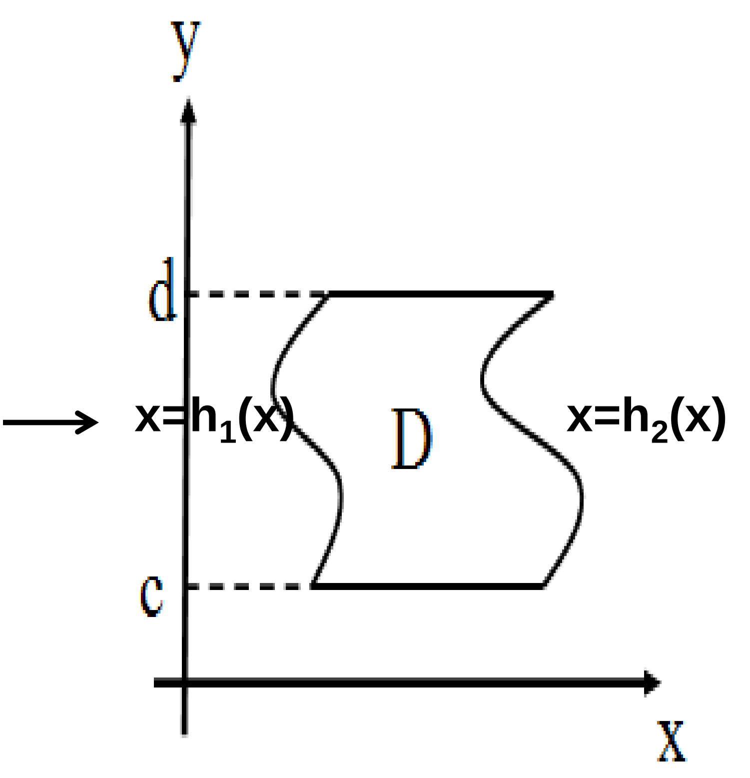
1.3.2 Type two

The plane region D is said to be type two if it lies between the graph of two continuous functions of y that is

$$D = \{(x,y)\, c \le y \le d, \quad h1(y) \le x \le h2(y)\}$$

$$\iint_D f(x,y)\,dA = \int_c^d \int_{h1(y)}^{h2(y)} f(x,y)\,dx\,dy$$





Example 1.3

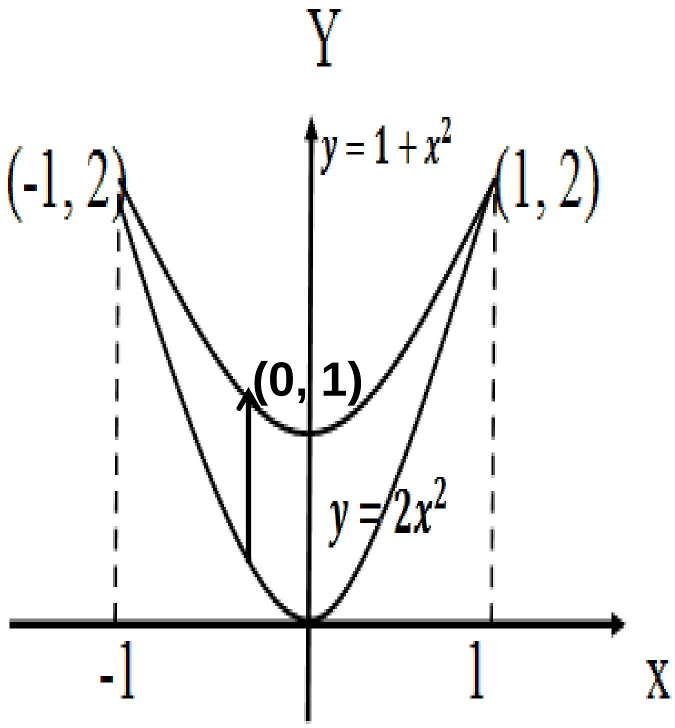
Evaluate $\iint_D (x+2y)\,dA$ **where D is the region bounded by parabolas**

Solution $y = 2x^2 \text{ and } y = 1 + x^2$

The intersection points of the two parabolas

$$2x^2 = 1 + x^2 \rightarrow x^2 = 1 \rightarrow x = \mp 1$$

$$D = \{(x,y) - 1 \le x \le 1, 2x^2 \le y \le 1 + x^2\}$$



$$\iint (x+2y)dA = \int_{-1}^{1}\int_{2x^2}^{1+x^2} (x+2y)\,dydx$$

$$= \int_{-1}^{1} [xy + y^2]_{2x^2}^{1+x^2} dx = \int_{-1}^{1} [x(1+x^2) + (1+x^2)^2 - x(2x^2) - (2x^2)]dx$$

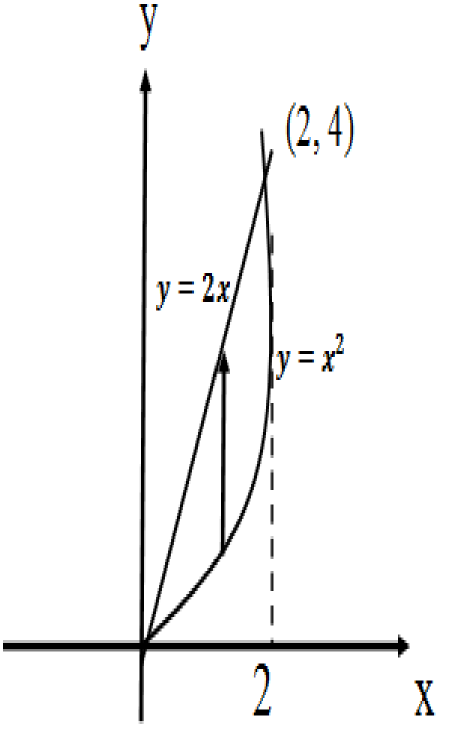
$$= \int_{-1}^{1} (-3x^4 - x^3 + 2x^2 + x + 1)dx = \frac{-3}{5}x^5 - \frac{x^4}{4} + \frac{2}{3}x^3 + \frac{x^2}{2} + x\,\Big]_{-1}^{1} = \frac{32}{15}$$

Example 1.4

Find the volume of the solid that lies under the parabola $z = x^2 + y^2$ and above the region D in the xy-plane bounded by the line y=2x and the parabola $y = x^2$

Solution

To find the intersection point



There are two ways to solve this probl

The first way

$$2x = x^2 \rightarrow x = 0 \ and \ x = 2$$

$$D = \{(x,y) 0 \leq x \leq 2, x^2 \leq y \leq 2x\}$$

$$V = \iint (x^2 + y^2) dA = \int_0^2 \int_{x^2}^{2x} (x^2 + y^2)\, dy dx$$

$$= \int_0^2 \left[x^2 y + \frac{y^3}{3}\right]_{x^2}^{2x} dx = \int_0^2 \left[x^2(2x) + \frac{(2x)^3}{3} - x^2 x^2 - \frac{(x^2)^3}{3}\right] dx$$

$$= \int_0^2 \left(\frac{-x^6}{3} + x^4 + \frac{14x^3}{3}\right) dx = -\frac{x^7}{21} - \frac{x^5}{5} + \frac{7x^4}{6}\Bigg]_0^2 = \frac{216}{35}$$

The second way

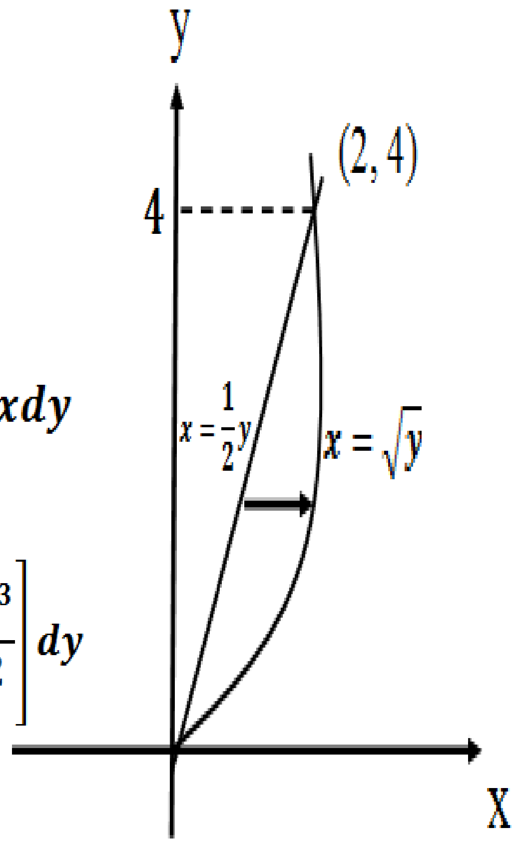
$$x = \frac{1}{2}y = \sqrt{y} \rightarrow y = 0 \ and \ y = 4$$

$$D = \left\{(x, y) 0 \le y \le 4, \frac{1}{2}y \le x \le \sqrt{y}\right\}$$

$$V = \iint_D (x^2 + y^2) dA = \int_0^4 \int_{\frac{1}{2}y}^{\sqrt{y}} (x^2 + y^2)\, dx dy$$

$$= \int_0^4 \left[\frac{x^3}{3} + y^2 x\right]_{\frac{1}{2}y}^{\sqrt{y}} dy = \int_0^4 \left[\frac{y^{\frac{3}{2}}}{3} + y^{\frac{5}{2}} - \frac{y^3}{24} - \frac{y^3}{2}\right] dy$$

$$= \frac{2}{15}y^{\frac{5}{2}} + \frac{2}{7}y^{\frac{7}{2}} - \frac{13}{96}y^4 \Big]_0^4 = \frac{216}{35}$$



Practices

1- Evaluate $\iint_D xy\,dA$ **where is the region bounded by y=x-1 and the parabola y=2x+6**

2- Find the volume of the tetrahedron bounded by the planes x+2y+z=0 , x=2y, x=0 and z=0

1.4 Reversing the order of integration

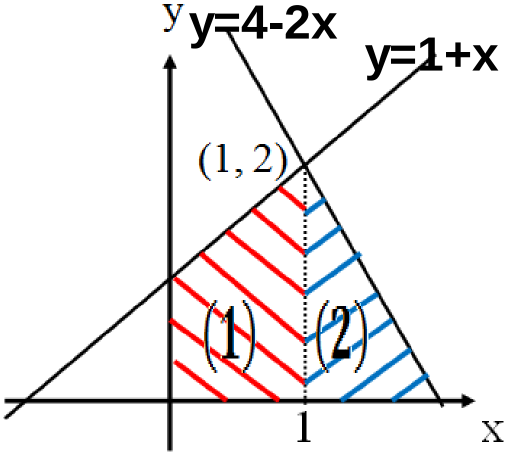
In some cases, it is necessary to reverse the order of integration and must be also change the boundary of it.

Example 1.5

Reverse the order of integration

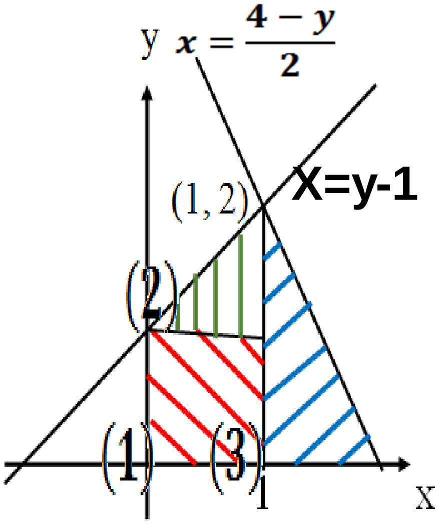
$$I=\int_{x=0}^{x=1}\int_{y=0}^{y=1+x} x\,dydx+\int_{x=1}^{x=2}\int_{y=0}^{y=4-2x} x\,dydx$$

We can divide the integration into two parts



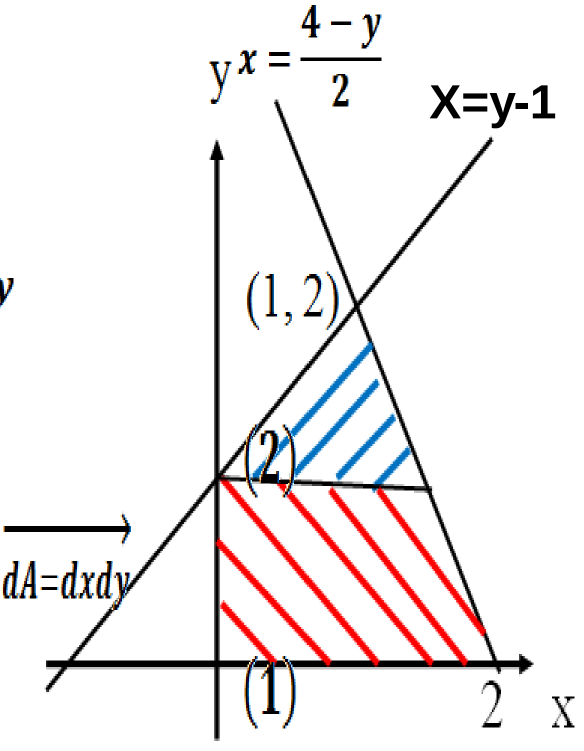
$$I=\int_{y=0}^{y=1}\int_{x=0}^{x=1} xdxdy+\int_{y=1}^{y=2}\int_{x=y-1}^{x=1} xdxdy\int_{y=0}^{y=2}\int_{x=1}^{x=\frac{4-y}{2}} xdxdy$$

$\longrightarrow$ $dA=dxdy$



Or

$$I=\int_{y=0}^{y=1}\int_{x=0}^{x=\frac{4-y}{2}} x\,dxdy+\int_{y=0}^{y=1}\int_{x=y-1}^{x=\frac{4-y}{2}} x\,dxdy$$



Practice

Reverse the order of integration

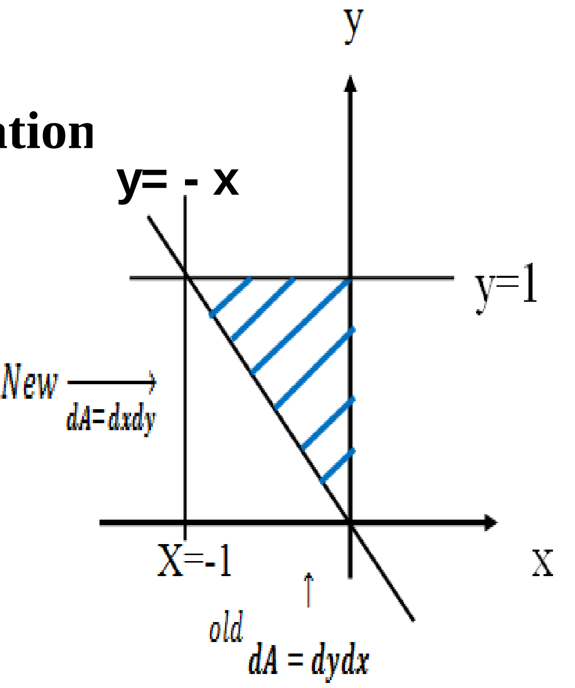
$$I=\int_{y=0}^{y=\frac{1}{\sqrt{2}}}\int_{x=0}^{x=\sin^{-1}y} dxdy+\int_{y=\frac{1}{\sqrt{2}}}^{y=1}\int_{x=0}^{x=\cos^{-1}y} dxdy$$

Example 1.6

Find $I=\int_{x=-1}^{x=0}\int_{y=-x}^{y=1}\frac{1}{y}\sin y\cos\frac{x}{y}\,dydx$

Solution

To reverse the order of integration



$$I = \int_{y=0}^{y=1} \int_{x=-y}^{x=0} \frac{1}{y} \sin y \cos\frac{x}{y}\, dx dy$$

$$= \int_{y=0}^{1} \sin y \sin\frac{x}{y}\Big]_{-y}^{0} dy$$

$$= \int_{y=0}^{1} [0 - \sin y \sin(-1)] dy = \int_{y=0}^{1} \sin(1) \sin y \, dy$$

$$= \sin(1)(-\cos y])_{0}^{1} = -\sin(1)\cos(1) + \sin(1) * 1 = 0.386 \; unit\ volume$$

Practice

Find

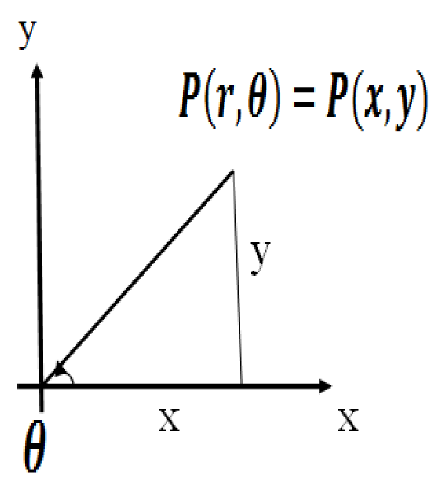
$$I = \int_{y=0}^{y=4} \int_{x=\sqrt{y}}^{x=2} e^{x^3} dx dy$$

1.4 Double integral in polar coordinators

Suppose that we want to evaluate a double integral $\iint_R f(x,y)\,dA$ where R is the region. From the figure we can find the relations between the polar coordinate (r, θ) and rectangular coordinate (x, y) by the equation.

$$r^2 = x^2 + y^2 \quad x = r\cos\theta, \qquad y = r\sin\theta$$

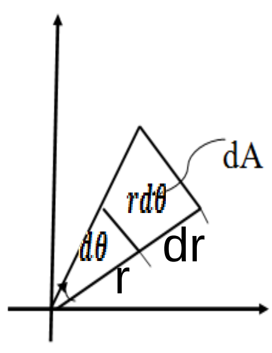
If f is continuous on a polar rectangle R g

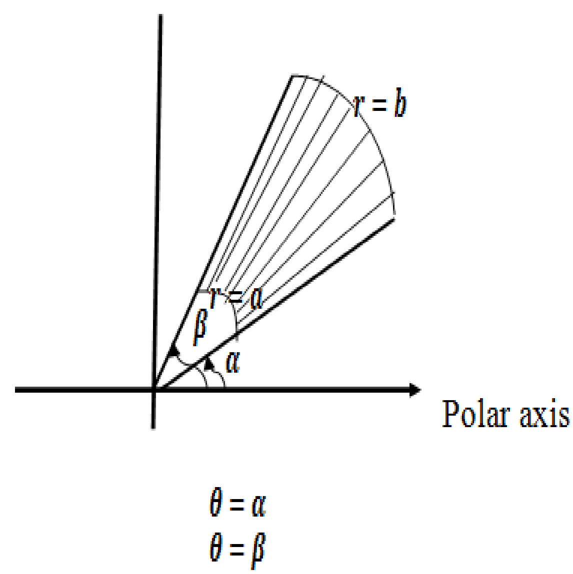


$$a \le r \le b, \qquad \alpha \le \theta \le \beta \text{ where } 0 \le \beta - \alpha \le 2\pi$$

$$\iint_R f(x,y)\,dA = \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta)\, r\,dr\,d\theta$$

$$dA = r\,dr\,d\theta$$





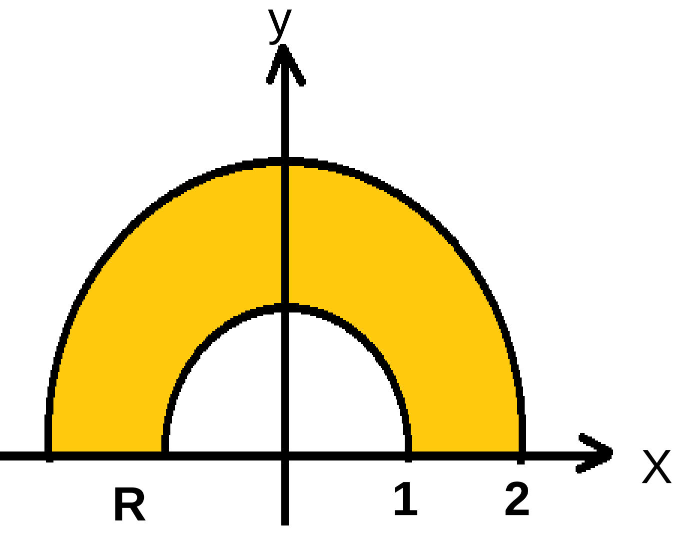
Example 1.7

Evaluate $\iint_R (3x + 4y^2)\,dA$ **where R is the region in the upper half-plane bounded by circle** $x^2 + y^2 = 1$ *and* $x^2 + y^2 = 4$

Solution

$$R = \{(x, y)\, y \geq 0, 1 \leq x^2 + y^2 \leq 4\}$$

$$R = \{(r, \theta)\, 0 \leq \theta \leq \pi, 1 \leq r \leq 2\}$$



$$\iint (3x + 4y^2)\,dA$$

$$= \int_0^{\pi}\int_1^{2} (3r\cos\theta + 4r^2\sin^2\theta)\, r\,dr\,d\theta$$

$$= \int_0^{\pi}\int_1^{2} (3r^2\cos\theta + 4r^3\sin^2\theta)\,dr\,d\theta$$

$$= \int_0^{\pi} r^3\cos\theta + r^4\sin^2\theta\Big]_1^2\, d\theta = \int_0^{\pi} (7\cos\theta + 15\sin^2\theta)\,d\theta$$

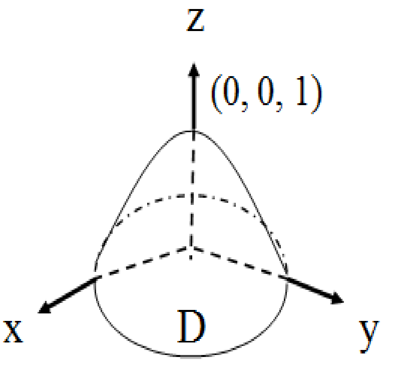
$$= \int_0^{\pi} \left(7\cos\theta + \frac{15}{2}(1 - \cos 2\theta)\right) d\theta$$

$$= \int_0^{\pi} \left(7\cos\theta + \frac{15}{2}(1 - \cos 2\theta)\right) d\theta = 7\sin\theta + \frac{15}{2}\theta - \frac{15}{4}\sin 2\theta\Big]_0^{\pi} = \frac{15\pi}{2}$$

Example 1.8

Find the volume of the solid bounded by the plane $z = 0$ and the parabola $z = 1 - x^2 - y^2$

Solution



$$when\; z = 0, x^2 + y^1 = 1$$

$$D = \{(r,\theta) 0 \leq \theta \leq 2\pi, 0 \leq r \leq 1\}$$

$$V = \int\int (1 - x^2 - y^2) dA = \int_0^{2\pi} \int_0^1 (1 - r^2) r dr d\theta$$

$$V = \int_0^{2\pi} d\theta \int_0^1 (r - r^3) dr = 2\pi \left[\frac{r^2}{2} - \frac{r^4}{4}\right] = \frac{\pi}{2} \; unit\; volume$$

1.5 Triple Integral

Triple integral can be defined for the three variables functions

$$I = \int\int\int_{G} dV$$

dV : volume of an element

dV= dz dA =dz dx dy = dz r dr dθ

G: bounded volume

Tripple integral can be solved as

$$I = \iint_{R} \left(\int_{z1}^{z2} dz\right) dA$$

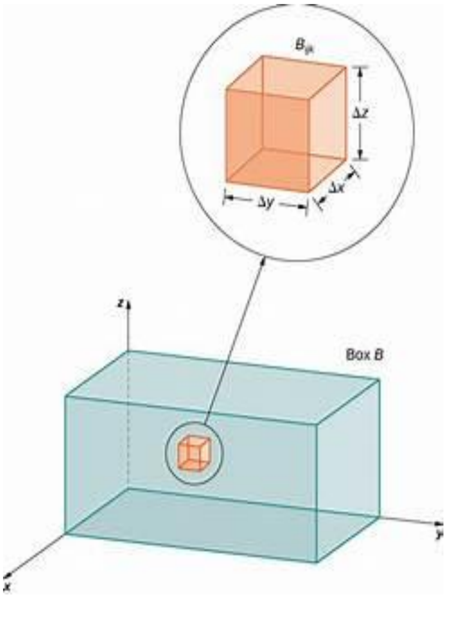
Triple integral can be classified into two types

1.5.1 Triple integral over a rectangular box

Which can designed as

$$I = \iiint_{B} dV = \int_{x=a}^{x=b}\int_{y=c}^{y=d}\int_{z=m}^{z=n} dzdxdy$$

$$B = \{(x,y,z)a \le x \le b, c \le y \le d, m \le z \le n\}$$



Example 1.9

Evaluate the triple integral $\iiint_B xyz^2 dV$ **where B is the rectangular box given by**

$$B = \{(x, y, z) 0 \le x \le 1, -1 \le y \le 2, 0 \le z \le 3\}$$

Solution

We could use any of the six possible order of integration if we choose to integrate with respect to x then y and then z as

$$I = \iiint_B xyz^2 dV = \int_{z=0}^{z=3} \int_{y=-1}^{y=2} \int_{x=0}^{x=1} xyz^2 dzdxdy$$

$$= \int_0^3 \int_{-1}^2 \left[\frac{x^2yz^2}{2}\right]_0^1 dydz = \int_0^3 \int_{-1}^2 \frac{yz^2}{2} dydz$$

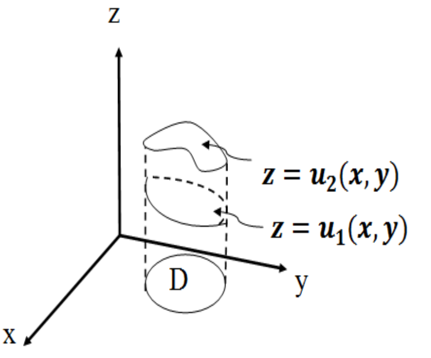
$$= \int_0^3 \left[\frac{y^2z^2}{4}\right]_{-1}^2 dz = \int_0^3 \frac{5z^2}{4} dz = \left.\frac{5z^3}{12}\right]_0^3 = \frac{135}{12}$$

1.5.2 Triple integration over non rectangular box

It can be defined an integral over a general bounded region E in the three dimensional space (a solid) by much the same procedure the used for double integral.

$$I = \iiint\limits_E f(x,y,z)\,dV$$

$$E = \{(x,y,z) \mid (x,y) \epsilon D, u_1(x,y) \leq z \leq u_2(x,y)\}$$

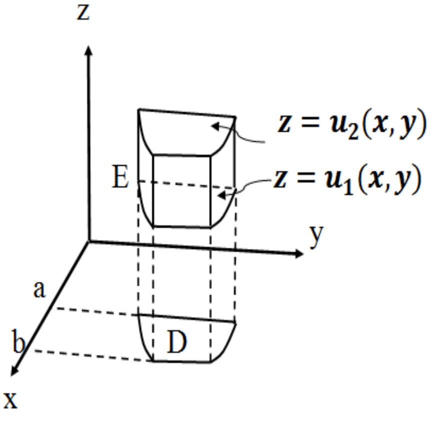


Where D is the projection of E on xy-plane as shown in the figure and notice that the upper boundary of the solid E in the surface equation $z = u_1(x,y)$ **and the lower boundary is the surface equation** $z = u_2(x,y)$

It can be shown that if E is a <u>type one</u> region given by the equation

$$\iiint\limits_E f(x,y,z)\,dV = \iint\limits_D \left[\int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\,dz\right] dA$$

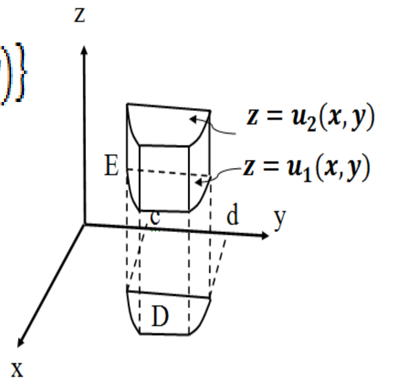
$$E = \{(x,y,z) \mid a \leq x \leq b, g_1(x) \leq y \leq g_2(x), u_1(x,y) \leq z \leq u_2(x,y)\}$$



$$\iiint\limits_E f(x,y,z)\,dV = \int_a^b \int_{g_1(x)}^{g_2(x)} \int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\,dz\,dy\,dx$$

If on the other hand, D is a <u>type two</u> plane region

$$E = \{(x,y,z) | c \le y \le d, h_1(y) \le x \le h_2(y), u_1(x,y) \le z \le u_2(x,y)\}$$

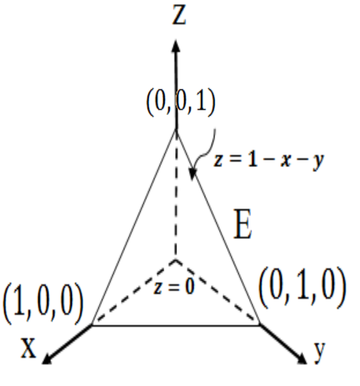


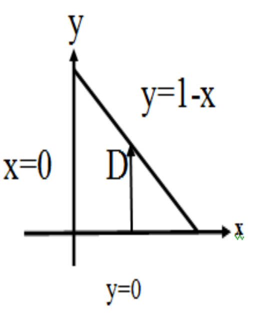
$$\iiint_E f(x,y,z)dV = \int_c^d \int_{h_1(y)}^{h_2(y)} \int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)dzdxdy$$

Example 1.10

Evaluate $\iiint_E zdV$ **where E is the solid tetrahedron bounded by four planes x=0, y=0, z=0 and x+y+z=1**

Solution





$$Upper\ Z = u_1(x,y) = 0$$

$$lower\ \ Z = u_2(x,y)$$

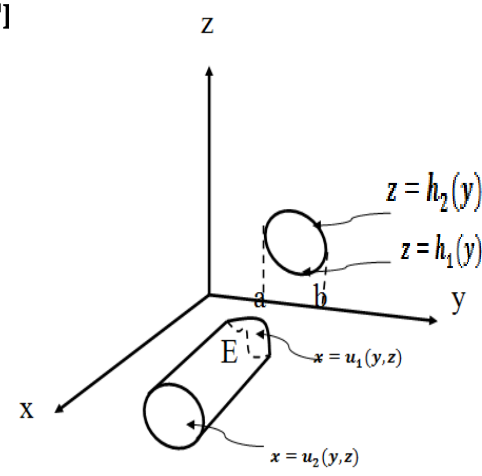
$$E = \{(x,y,z) | 0 \le x \le 1, 0 \le y \le 1 - x, 0 \le z \le 1 - x - y\}$$

$$\iiint_E zdV = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} zdzdxdy = \int_0^1 \int_0^{1-x} \frac{z^2}{2}\Big|_0^{1-x-y} dydx$$

$$= \frac{1}{2}\int_0^1 \int_0^{1-x} (1-x-y)^2 dydx = \frac{1}{2}\int_0^1 \left[-\frac{(1-x-y)^3}{3}\right]_0^{1-x} dx$$

$$= \frac{1}{6}\int_0^1 (1-x)^3 dx = \frac{1}{6}\left[-\frac{(1-x)^4}{4}\right]_0^1 = \frac{1}{24}\ unit\ volume$$

The solid region E maybe take other for



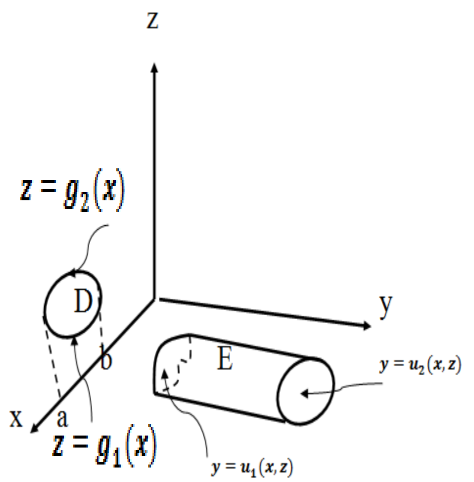
$$E = \{((x,y,z)|((y,z)\epsilon D, u_1(y,z) \leq x \leq u_2(y,z)\}$$

$$\iiint_E f(x,y,z)dV = \iint_D \left[\int_{u_1(y,z)}^{u_2(y,z)} f(x,y,z)dx\right] dA$$

$$= \int_c^d \int_{h1(y)}^{h2(y)} \int_{u1(y,z)}^{u2(y,z)} f(x,y,z)dxdzdy$$

And also

$$E = \{((x,y,z)|((x,z)\epsilon D, u_1(x,z) \leq y \leq u_2(x,z)\}$$



$$\iiint_E f(x,y,z)dV = \iint_D \left[\int_{u_1(x,z)}^{u_2(x,z)} f(x,y,z)dy\right] dA$$

$$= \int_a^b \int_{g1(x)}^{g2(x)} \int_{u1(x,z)}^{u2(x,z)} f(x,y,z)dydzdx$$

Example 1.11

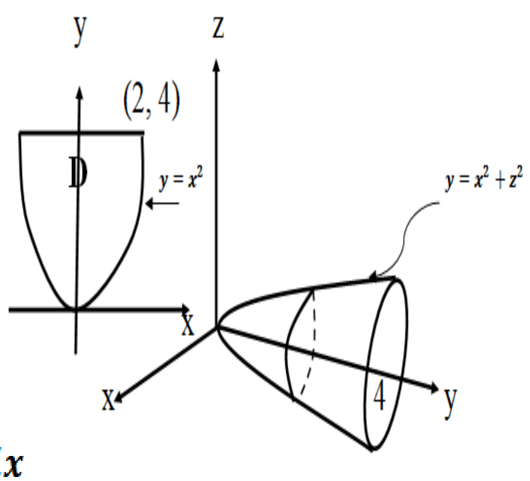
Evaluate $\iiint_E \sqrt{x^2+z^2}\,dV$ **where E is the region bounded by the parabolic** $y = x^2 + y^2$ and the plane $y = 4$

Solution

It can consider D onto the xy-plane

Z=0 , $y = x^2$

From



$$y = x^2 + y^2 \rightarrow z = \pm\sqrt{y - x^2}$$

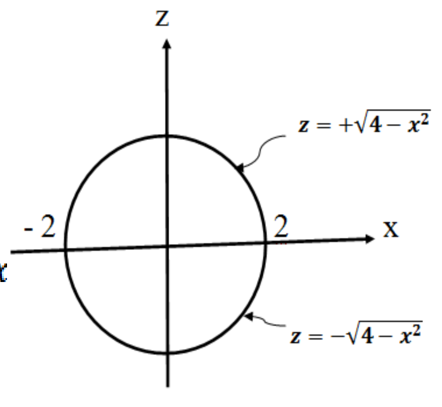
$$\forall = \iiint_E \sqrt{y - x^2}\,dV = \int_{-2}^{2}\int_{x^2}^{4}\int_{-\sqrt{y-x^2}}^{\sqrt{y-x^2}} \sqrt{y - x^2}\,dz\,dy\,dx$$

It hard to integrate

The projection plane can be changed

$$\forall = \iiint_E \sqrt{x^2+z^2}\,dV = \iint_D \left[\int_{x^2+z^2}^{4} \sqrt{x^2+z^2}\,dy\right] dA$$

$$= \iint_D (4 - x^2 - z^2)\sqrt{x^2+z^2}\,dA = \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (4 - x^2 - z^2)\sqrt{x^2+z^2}\,dz\,dx$$



It also hard to integrate

It can convert to polar coordinate in xz-plane

$$x = r\cos\theta,\quad z = r\sin\theta$$

$$\forall = \iint_D (4 - x^2 - z^2)\sqrt{x^2+z^2}\,dA = \int_0^{2\pi}\int_0^{2} (4 - r^2)\,r\,r\,dr\,d\theta$$

$$\forall = \int_0^{2\pi} d\theta \int_0^{2} (4r^2 - r^4)\,dr\,d\theta = 2\pi\left[\frac{4r^3}{3} - \frac{r^5}{5}\right]_0^2 = \frac{128}{15}\pi$$

1.6 Triple integral in cylindrical coordinates

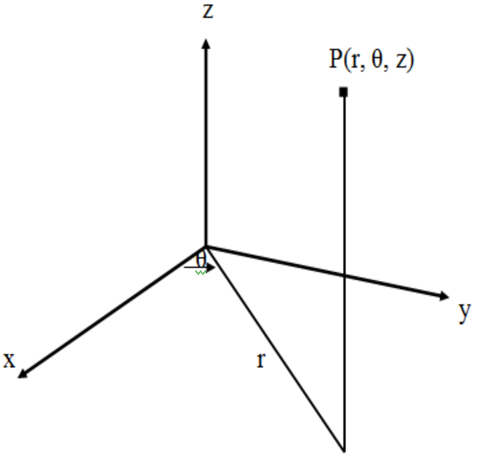
Let point P is represented in Cartesian coordinates as (x,y,z) and it can be represented in cylindrical coordinates as (r,θ, z)

To convert from cylindrical to rectangular coordinates

$$x = r\cos\theta \qquad y = r\sin\theta \qquad z = z$$

While, to convert from rectangular to cylindrical

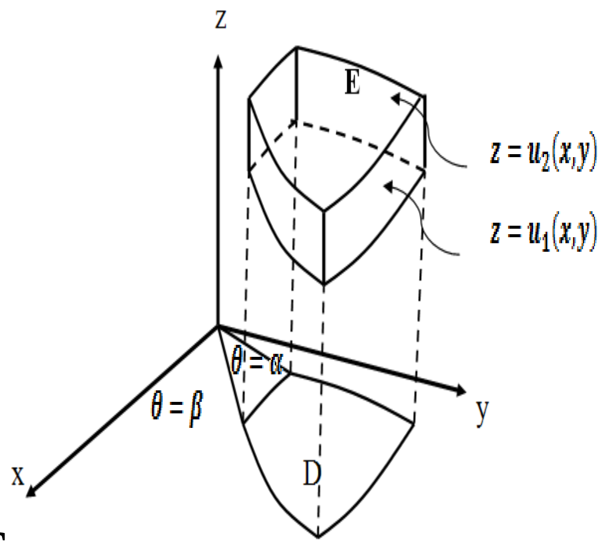
$$r = \sqrt{x^2 + y^2} \qquad \theta = \tan^{-1}\frac{y}{x} \qquad z = z$$



1.6.1 Evaluating Triple Integral with cylindrical coordinates

Suppose E is a type one region whose projection D on the xy-plane is conveniently described in polar coordinates as shown

$$E = \{(x, y, z) | (x, y) \epsilon D, u_1(x, y) \leq z \leq u_2(x, y)\}$$



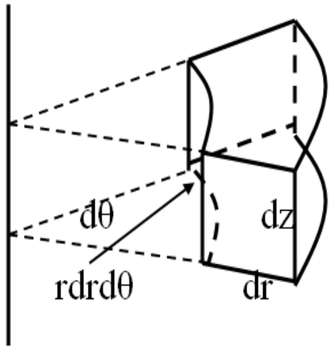
Where D is

$$D = \{(r, \theta) |, \alpha \leq \theta \leq \beta, h_1(\theta) \leq r \leq h_2(\theta)\}$$

To find the volume

$$\forall = \iiint_E f(x, y, z) dV = \iint_D \left[\int_{u_1(x,y)}^{u_2(x,y)} f(x, y, z)\right] dA$$

But, it can be evaluated by polar c



$$dV = r dr d\theta dz$$

$$\forall = \iiint_E f(x, y, z) dV = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} \int_{u_1(r\cos\theta, r\sin\theta)}^{u_2(r\cos\theta, r\sin\theta)} f(r\cos\theta, r\sin\theta, z) r dz dr d\theta$$

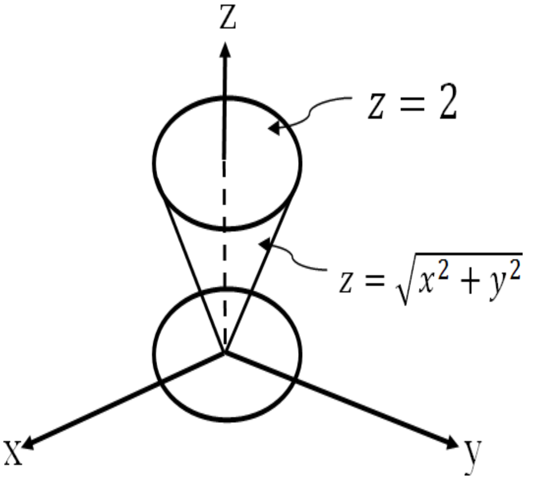
Example 1.12

Evaluate $\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{\sqrt{x^2+y^2}}^{2}(x^2+y^2)\,dz\,dy\,dx$

Solution

$$E=\left\{(x,y,z)\,\middle|\,-2\le x\le 2,-\sqrt{4-x^2}\le y\le\sqrt{4-x^2},\sqrt{x^2+y^2}\le z\le 2\right\}$$

The region has much simpler describing in Cylindrical coordinates



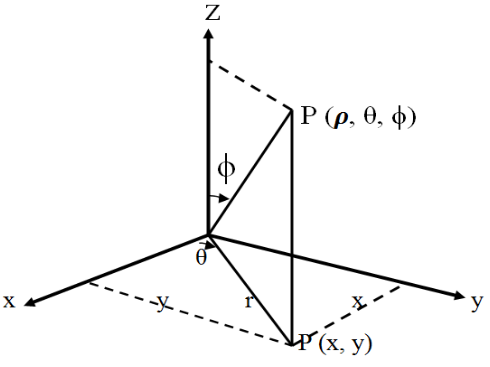
$$E=\left\{(r,\theta,z)\,\middle|\,0\le\theta\le 2\pi,0\le y\le 2,r\le z\le 2\right\}$$

$$I=\iiint_E(x^2+y^2)dV=\int_0^{2\pi}\int_0^2\int_r^2 r^2 r\,dz\,dr\,z\,d\theta$$

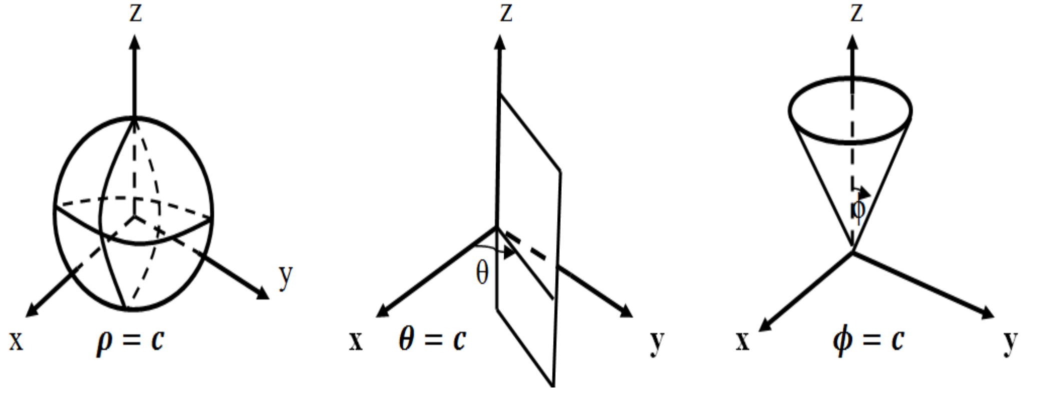
$$=\int_0^{2\pi}d\theta\int_0^2 r^3(2-r)dr=2\pi\left[\frac{1}{2}r^2-\frac{1}{5}r^2\right]_0^2=\frac{16}{5}\pi$$

1.7 Triple Integral in Spherical Coordinates

Another useful coordinate system in three dimensions the spherical coordinate system as shown in the figure. It simplifies the evaluation of triple integrals over region bounded by sphere or cone.



The spherical coordinates system is especially useful in problems where is symmetry about a point and the origin is placed at this point as



The relationship between rectangular and spherical coordinates according to figure above

$$z = \rho cos\emptyset \quad r = \rho sin\emptyset \quad but\ x = rcos\theta \quad y = rsin\theta$$

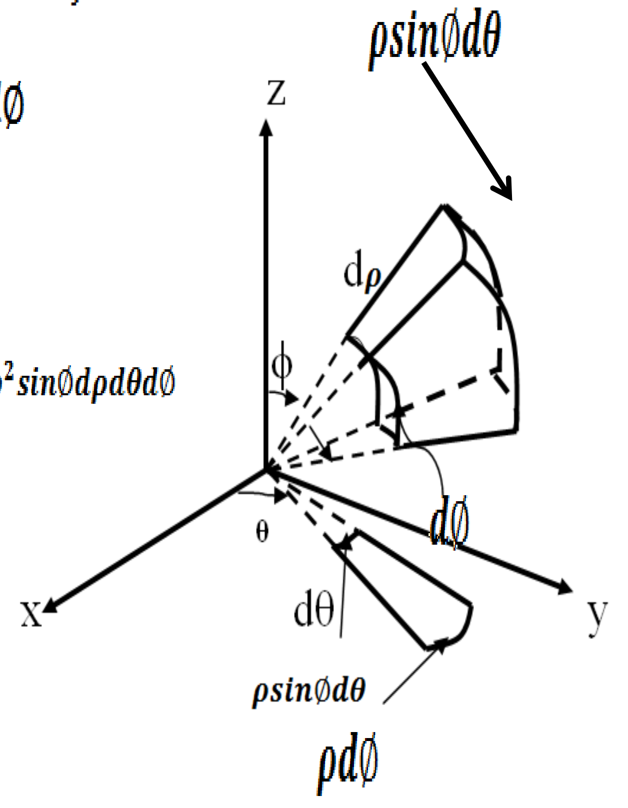
So to convert from spherical to rectangular coordinates

$$x = \rho sin\emptyset cos\theta, \quad y = \rho sin\emptyset sin\theta. \quad z = \rho cos\emptyset, \qquad \rho^2 = x^2 + y^2 + z^2$$

1.7.1 Evaluate triple Integrals with Spherical Coordinates

$$E = \{(\rho, \theta, \emptyset) |, a \le \rho \le b, \alpha \le \theta \le \beta, c \le \emptyset \le d\}$$

$$dV = d\rho(\rho d\emptyset)(\rho sin\emptyset d\theta) = \rho^2 sin\emptyset d\rho d\theta d\emptyset$$



$$\iiint_E f(x,y,z)dV$$

$$= \int_c^d \int_\alpha^\beta \int_a^b f(\rho sin\emptyset cos\theta, \rho sin\emptyset sin\theta, \rho sin\emptyset)\, \rho^2 sin\emptyset d\rho d\theta d\emptyset$$

Example 1.13

Evaluate $\iiint_B e^{(x^2+y^2+z^2)^{\frac{2}{3}}} dV$ **w**

Solution

$$B = \{(x.y.z)|, x^2 + y^2 + z^2 \le 1\}$$

$$B = \{(\rho, \theta, \emptyset)|, 0 \le \rho \le 1, 0 \le \theta \le 2\pi, 0 \le \emptyset \le \pi\}$$

$$\rho^2 = x^2 + y^2 + z^2$$

$$\iiint_B e^{(x^2+y^2+z^2)^{\frac{2}{3}}} dV = \int_0^{\pi} \int_0^{2\pi} \int_0^1 e^{(\rho^2)^{\frac{3}{2}}} \rho^2 sin\emptyset d\rho d\theta d\emptyset$$

$$= \int_0^{\pi} \sin\emptyset \, d\emptyset \int_0^{2\pi} d\theta \int_0^{1} \rho^2 e^{\rho^3} d\rho$$

$$= [-\cos\emptyset]_0^{\pi} * 2\pi \left[\frac{1}{3} e^{\rho^3}\right]_0^1 = 2.3\pi \; unit \; volume$$

Practice

Use spherical coordinates to find the volume of the solid that lies about the cone $z = \sqrt{x^2 + y^2}$ **and below the sphere** $x^2 + y^2 + z^2 = 8$

Hints: Equations of some geometrical shapes

$Sphare: \; (x-h)^2 + (y-k)^2 + (z-l)^2 = a^2$ $\quad h, k, l$ are the center and a is thr radius

$$Ellipsoid: \; \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \quad (a > b > c)$$

$$Paraboloid: \; z = \frac{x^2}{a^2} + \frac{y^2}{b^2}$$

$$Cone: \; z = x^2 + y^2$$

$Cylinder: \; x^2 + y^2 = a^2 \text{ for all } z \quad x^2 + z^2 = a^2 \text{ for all } y \quad y^2 + z^2 = a^2 \text{ for all } x$

Sheet No (1)

1- Evaluate

$$(a)\int_0^3\int_0^1 1+4xy\,dxdy \qquad (b)\int_0^2\int_0^\pi r\sin^2\theta\,d\theta dr \qquad (c)\int_1^4\int_1^2\left(\frac{x}{y}+\frac{y}{x}\right)dydx$$

2- Calculate the following double integrals

$$(a)\iint_R (6x^2y^3-5y^4)dA \quad R=\{(x,y)|0\le x\le 3, 0\le y\le 1\}$$

$$(b)\iint_R \cos(x+2y)dA \quad R=\left\{(x,y)|0\le x\le \pi, 0\le y\le \frac{\pi}{2}\right\}$$

$$(c)\iint_R x\sin(x+y)dA \quad R=\left[0,\frac{\pi}{2}\right]*\left[0,\frac{\pi}{3}\right]$$

3 (a) Find the volume of the solid that lies under the plane $3x+2y+z=12$ and above rectangular $R=\left\{(x,y)|0\le x\le 1, 0\le y\le \frac{\pi}{2}\right\}$

(b) Find the volume of the solid in the first octant bounded by the cylinder $z=16-x^2$ and the plane y=5

(4) Evaluate

$$(a)\int_0^4\int_0^{\sqrt{y}} xy^2\,dxdy \qquad (b)\int_0^1\int_{x^2}^{x}(1+2y)\,dydx$$

5- Evaluate the double integrals

$$(a)\iint_D x\,dA \quad D=\{(x,y)|0\le x\le \pi, 0\le y\le \sin x\}$$

$$(b)\iint_D y^2e^{xy}dA \quad D=\{(x,y)|0\le y\le 4, 0\le x\le y\}$$

6- Sketch the region of integration and change the order of integration

$$(a)\int_0^4\int_0^{\sqrt{x}} f(x,y)\,dydx \quad (b)\int_0^3\int_0^{\sqrt{4-y}} f(x,y)dxdy \quad (c)\int_0^1\int_{3y}^3 e^{x^2}\,dxdy \quad (d)\int_0^1\int_x^1 e^{\frac{x}{y}}dydx$$

7- Find the volume of the solid that lies under the paraboloid $x^2+y^2=2x$ above the xy-plane and inside the cylinder $z=x^2+y^2$

8- Evaluate the triple integrals

$$(a)\int_0^1\int_0^z\int_0^{x+z} 6xz\,dydxdz \quad (b)\int_0^{\sqrt{\pi}}\int_0^x\int_0^{xz} x^2 \sin y\,dydzdx$$

9- Use the triple integral to find the volume of the given solid

(a) The tetrahedron enclosed by the coordinate planes and plane $2x+y+z=4$

(b) The solid bounded by the cylinder $y=x^2$ and the plane $z=0, z=4 \text{ and } y=9$

10- Evaluate $\iiint_E \sqrt{x^2+y^2}\,dV$ where E is the region that lies inside the cylinder $x^2+y^2=16$ and between $z=-5 \text{ and } z=4$

11- Find the volume of the solid that lies within both the cylinder $x^2+y^2=1$ and the sphere $x^2+y^2+z^2=4$

12- Evaluate $\iiint_B \left(x^2+y^2+z^2\right)^2 dV$ where B is a ball with center the origin and radius 5.

13-Evaluate $\iiint_E z\,dV$ where E lies between the spheres $x^2+y^2+z^2=1$ and $x^2+y^2+z^2=4$ in the first octant

Chapter two

Laplace Transform

$$\mathscr{L}\{f(t)\} = F(s)$$

2.1 Introduction

The Laplace transform takes a function of time and transforms it to a function of a complex variable s. Because the transform is invertible, no information is lost and it is reasonable to think of a function f(t) and its Laplace transform F(s) as two views of the same phenomenon. Each view has its uses and some features of the phenomenon are easier to understand in one view or the other.

The Laplace transform of a function f(t) is defined by the integral

$$F(s) = \int_0^{\infty} e^{-st} f(t)dt$$

Also it can be written as

$$F(S) = \mathcal{L}[f(t)]$$

This integration is called <u>Improper</u> but mathematically can be represented as

$$\int_0^{\infty} e^{-st} f(t)dt = \lim_{h\to\infty}\left[\int_0^{h} e^{-st} f(t)dt\right]$$

2.2 Laplace transform for some functions

The most function is the exponential (e^{at}) and to transform it by Laplace $\mathcal{L}[e^{at}]$.

$$\mathcal{L}[e^{at}] = \int_0^\infty e^{-st} f(t)dt = \int_0^\infty e^{-st} e^{at} dt$$

$$= \int_0^\infty e^{-(s-a)} dt = -\frac{1}{s-a} e^{-(s-a)t}\Big|_0^\infty = -\frac{1}{s-a}[e^{-\infty} - e^0] = \frac{1}{s-a}$$

$$\mathcal{L}[e^{at}] = \frac{1}{s-a} \quad \textit{This is valid for } s > a$$

<u>Is the special case</u>

$a = 0$ *then*

$$e^{0t} = e^0 = 1$$

$$\mathcal{L}[1] = \frac{1}{s-0} = \frac{1}{s} \qquad \mathcal{L}[1] = \frac{1}{s}, \quad s > a$$

Another important function (t^n)

$\mathcal{L}[t^n] \quad n > 0$

$$\mathcal{L}[t^n] = \int_0^\infty e^{-st} f(t)dt = \int_0^\infty e^{-st} t^n dt$$

Integration by parts, we get

$$udv = uv - \int vdu$$

$$t^n\left(-\frac{1}{s}e^{-st}\right)\Big|_0^\infty - \int_0^\infty \left(-\frac{1}{s}e^{-st}\right) nt^n dt$$

$$-t^n \frac{1}{s} e^{-st}\Big|_0^\infty + \frac{n}{s}\int_0^\infty e^{-st} t^{n-1} dt$$

$$0-0+\frac{n}{s}\mathcal{L}[t^{n-1}]$$

$$\mathcal{L}[t^n]=\frac{n}{s}\mathcal{L}[t^{n-1}]$$

Keep going !

$$\mathcal{L}[t^n]=\frac{n}{s}\mathcal{L}[t^{n-1}]=\frac{n(n-1)}{s^2}\mathcal{L}[t^{n-2}]$$

$$\mathcal{L}[t^n]=\frac{n}{s}\mathcal{L}[t^{n-1}]=\frac{n(n-1)}{s^2}\mathcal{L}[t^{n-2}]=\cdots=\frac{n!}{s^n}\mathcal{L}[1]$$

$$\boxed{\mathcal{L}[t^n]=\frac{n!}{s^{n+1}} \quad n>0}$$

- **Let find** $cosat$

$$\mathcal{L}[cosat]=\int_0^\infty e^{-st}f(t)dt=\int_0^\infty e^{-st}\,cosatdt$$

Integrate by parts

$$udv=uv-\int vdu$$

$$e^{-st}\frac{1}{a}\sin at|_0^\infty-\int_0^\infty\frac{1}{a}\sin at(-se^{-st})\,dt$$

$$\left[\frac{1}{a}e^{-\infty}sin\infty\right]-\left[\frac{1}{a}e^0sin0\right]+\frac{s}{a}\int_0^\infty e^{-st}\sin at\,dt$$

$$\left[-t^\infty\frac{1}{s}\,e^{-\infty}\right]-\left[-0^n\frac{1}{s}\,e^0\right]+\frac{n}{s}\mathcal{L}[t^{n-1}]$$

$$0-0+\frac{s}{a}\int_0^{\infty} e^{-st}\sin at\,dt$$

$$\mathcal{L}[cosat]=\frac{s}{a}\int_0^{\infty} e^{-st}\sin at\,dt$$

Another integration by parts

$$\int_0^{\infty} e^{-st}\sin at\,dt = e^{-st}(-\frac{1}{a}\cos at)|_0^{\infty} - \int_0^{\infty} -\frac{1}{a}\cos at(-se^{-st})\,dt$$

$$=\left[\frac{1}{a}e^{-\infty}cos\infty\right]-\left[-\frac{1}{a}e^{0}cos0\right]-\frac{s}{a}\int_0^{\infty} e^{-st}\cos at\,dt$$

$$\mathcal{L}[cosat]=\frac{s}{a}\left[\frac{1}{a}-\frac{s}{a}\mathcal{L}[cosat]\right]$$

$$\mathcal{L}[cosat]=\frac{s}{a^2}-\frac{s^2}{a^2}\mathcal{L}[cosat]$$

$$\mathcal{L}[cosat]+\frac{s^2}{a^2}\mathcal{L}[cosat]=\frac{s}{a^2}$$

$$\mathcal{L}[cosat]\left(1+\frac{s^2}{a^2}\right)=\frac{s}{a^2}$$

$$[cosat]=\frac{\frac{s}{a^2}}{\left(1+\frac{s^2}{a^2}\right)}=\frac{s}{a^2\left(1+\frac{s^2}{a^2}\right)}$$

$$\mathcal{L}[cosat]=\frac{s}{(s^2+a^2)} \quad ,s>0$$

Practice

$$\mathcal{L}[sin\ at]$$

Table of some essential transforms

Function f(t)	Laplace transform(F(s)
e^{at}	$\mathcal{L}[e^{at}] = \frac{1}{s-a}$
1	$\mathcal{L}[1] = \frac{1}{s}$
t^n	$\mathcal{L}[t^n] = \frac{n!}{s^{n+1}}$
cos at	$\mathcal{L}[cos\ at] = \frac{s}{s^2+a^2}$
sin at	$\mathcal{L}[sin\ at] = \frac{a}{s^2+a^2}$

Example 2.1

Transform the function t^3 by Laplace.

Solution

$$\mathcal{L}[f(t)] = \mathcal{L}[t^3] = \frac{3!}{s^{3+1}} = \frac{3*2*1}{s^4} = \frac{6}{s^4} = F(s)$$

Example 2.2

Transform $f(t) = sin\ 2t$ to the F(s)

Solution

$$\mathcal{L}[f(t)] = \mathcal{L}[sin\ 2t] = \frac{a}{s^2+a^2} = \frac{2}{s^2+2^2} = \frac{2}{s^2+4} = F(s)$$

Example 2.3

Transform the function $f(t) = 5e^{2t} - t^4$ by Laplace.

Solution

$$\mathcal{L}[f(t)] = \mathcal{L}[5e^{2t} - t^4] = \int_0^\infty e^{-st}(5e^{2t} - t^4)dt$$

$$= \int_0^\infty e^{-st}(5e^{2t})dt - \int_0^\infty e^{-st}(t^4)dt$$

$$= \mathcal{L}[5e^{2t}] - \mathcal{L}[t^4] = 5\mathcal{L}[e^{2t}] - \mathcal{L}[t^4]$$

$$= 5 * \frac{1}{s-2} - \frac{4!}{S^5} = \frac{5}{s-2} - \frac{24}{s^5} = F(s)$$

Example 2.4

Transform $f(t) = t^3 - 5 + cos\ 2t$ to the F(s)

Solution

$$\mathcal{L}[f(t)] = \mathcal{L}[t^3 - 5 + cos\ 2t]$$

$$= \mathcal{L}[t^2] - \mathcal{L}[5] + \mathcal{L}[cos\ 2t]$$

$$\mathcal{L}[f(t)] = \frac{2!}{s^{2+1}} - 5 * \frac{1}{s} + \frac{s}{s^2 + 2^2}$$

$$F(s) = \frac{2}{s^3} - \frac{5}{s} + \frac{s}{s^2 + 4}$$

Practice Transform these functions by laplace

$$(a)\ f(t) = t^3 + e^{2t} + sin\ 5t \quad (b)\ e^{-2t} - 14t$$

2.3 Inverse Laplace transform

As it is known, the main purpose of Laplace transform is to convert the regular function into a new function could be solved easily. The result also could be converted back to the regular form as.

$$f(t) \xrightarrow{\mathcal{L}[f(t)]} F(s) \xrightarrow{\mathcal{L}^{-1}[F(s)]} f(t)$$

The table of Laplace transform can be added to it inverse column.

Function f(t)	Laplace transform F(s)	Inverse Laplace transform f(t)
e^{at}	$\mathcal{L}[e^{at}] = \frac{1}{s-a}$	$\mathcal{L}^{-1}\left[\frac{1}{s-a}\right] = e^{at}$
1	$\mathcal{L}[1] = \frac{1}{s}$	$\mathcal{L}^{-1}\left[\frac{1}{s}\right] = 1$
t^n	$\mathcal{L}[t^n] = \frac{n!}{s^{n+1}}$	$\mathcal{L}^{-1}\left[\frac{1}{s^n}\right] = \frac{t^{n-1}}{(n-1)!}$
cos at	$\mathcal{L}[\cos at] = \frac{s}{s^2+a^2}$	$\mathcal{L}^{-1}\left[\frac{s}{s^2+a^2}\right] = \cos at$
sin at	$\mathcal{L}[\sin at] = \frac{a}{s^2+a^2}$	$\mathcal{L}^{-1}\left[\frac{1}{s^2+a^2}\right] = \frac{1}{a}\sin at$

Example 2.5

Find the inverse transform of $F(s) = \frac{1}{s-2}$

Solution

$$f(t) = e^{2t}$$

Example 2.6

Find the inverse transform of $F(s) = \frac{1}{2s+1}$

Solution

$$F(s) = \frac{1}{2s+1} = \frac{1}{2s-(-1)} = \frac{1}{2\left(s-\left(-\frac{1}{2}\right)\right)}$$

$$f(t) = \frac{1}{2}\mathcal{L}^{-1}\left[\frac{1}{s-\left(-\frac{1}{2}\right)}\right] = \frac{1}{2}e^{-\frac{1}{2}t}$$

Example 2.7

Find the inverse transform of $F(s) = \frac{2}{s^4}$

Solution

$$f(t) = 2\mathcal{L}^{-1}\left[\frac{1}{s^4}\right] = 2\frac{t^3}{3*2*1} = \frac{t^3}{3}$$

Example 2.7

Find the inverse transform of $F(s) = \frac{2}{s^2+3}$

Solution

$$f(t) = 2\mathcal{L}^{-1}\left[\frac{1}{s^2+\left(\sqrt{3}\right)^2}\right] = \frac{2}{\sqrt{3}}\sin\sqrt{3}\,t$$

Practice: Find $F(s) = \mathcal{L}^{-1}\left[\frac{4}{s+2} - \frac{6}{s^2+3} + \frac{1}{s^5}\right]$

2.3 Differentiation property of Laplace transform

To find Laplace of deferential function as

$$\mathcal{L}[x^{/}(t)] = \mathcal{L}\left[\frac{dx}{dt}\right] = \mathcal{L}[D_x]$$

That should be

$$\mathcal{L}[x^{/}(t)] = \int_0^{\infty} e^{-st} x^{/}(t) dt$$

By integration of parts,

$$\mathcal{L}[x^{/}(t)] = \mathcal{L}[D_x] = s\mathcal{L}[x(t)] - x(0)$$

$$\mathcal{L}[x^{//}(t)] = \mathcal{L}[{D_x}^2] = s^2\mathcal{L}[x(t)] - sx(0) - x^{/}(0)$$

$$\mathcal{L}[{D_x}^3] = s^3\mathcal{L}[x(t)] - s^2x(0) - sx^{/}(0) - x^{//}(0)$$

Example 2.8

Solve the differential equation by Laplace transform

$$\frac{dx}{dt} = t \quad , x(0) = 2$$

Solution

$$\frac{dx}{dt} = D_x = x^{/}(t) = t$$

Bt take Laplace for both side

$$\mathcal{L}[D_x] = \mathcal{L}[t]$$

$$s\mathcal{L}[x(t)] - x(0) = \frac{1!}{s^{1+1}}$$

$$s\mathcal{L}[x(t)] - 2 = \frac{1}{s^2}$$

$$\mathcal{L}[x(t)] = \frac{1}{s^3} + \frac{2}{s}$$

$$x(t) = \mathcal{L}^{-1}\left[\frac{1}{s^3} + \frac{2}{s}\right]$$

$$x(t) = \frac{t^{3-1}}{(3-1)!} + 2 * 1$$

$$x(t) = \frac{t^2}{2} + 2$$

Example 2.9

Solve the following differential equation by using Laplace transform

$$D^3x - D^2x = 0 \quad , x(0) = x^{/}(0) = x^{//}(0) = 3$$

Solution

$$D^3x - D^2x = 0$$

By taking Laplace for this equation

$$\mathcal{L}[D^3x] - \mathcal{L}[D_x{}^2] = \mathcal{L}[0]$$

$$\left(s^3\mathcal{L}[x(t)] - s^2x(0) - sx^{/}(0) - x^{//}(0)\right) - \left(s^2\mathcal{L}[x(t)] - sx(0) - x^{/}(0)\right) = 0$$

$$s^3\mathcal{L}[x(t)] - 3s^2 - 3s - 3 - s^2\mathcal{L}[x(t)] + 3s + 3 = 0$$

$$s^3\mathcal{L}[x(t)] - s^2\mathcal{L}[x(t)] - 3s^2 = 0$$

$$\mathcal{L}[x(t)](s^3 - s^2) = 3s^2$$

$$\mathcal{L}[x(t)] = \frac{3s^2}{s^3 - s^2}$$

$$\mathcal{L}[x(t)] = \frac{3}{s-1}$$

$$x(t) = \mathcal{L}^{-1}\left[\frac{1}{s-1}\right]$$

$$x(t) = 3e^t$$

Example 2.10

Solve the following differential equation by using Laplace transform.

$$x^{//} - x = e^{2t} \quad , x(0) = 0, x^{/}(0) = 1$$

Solution

$$y^{//} - y = e^{2t}$$

$$s^2\mathcal{L}[x(t)] - sx(0) - x^{/}(0) - \mathcal{L}[x(t)] - x(0) = \frac{1}{s-2}$$

$$s^2\mathcal{L}[x(t)] - 0 - 1 - \mathcal{L}[x(t)] - 0 = \frac{1}{s-2}$$

$$\mathcal{L}[x(t)](s^2 - 1) = \frac{s-1}{s-2}$$

$$\mathcal{L}[x(t)] = \frac{s-1}{(s-2)(s^2-1)} = \frac{1}{(s-2)(s+1)}$$

$$\frac{1}{(s-2)(s+1)} = \frac{A}{s-2} + \frac{B}{s+1}$$

$$As + A + Bs - 2B = 1$$

$$At\ S = -1, \qquad 1 = A(0) + B(-3) \rightarrow B = -\frac{1}{3}$$

$$At\ S = 2, \qquad 1 = A(3) + B(0) \rightarrow A = \frac{1}{3}$$

$$\frac{1}{(s-2)(s+1)} = \frac{\frac{1}{3}}{s-2} - \frac{\frac{1}{3}}{s+1}$$

$$\mathcal{L}[x(t)] = \frac{\frac{1}{3}}{s-2} - \frac{\frac{1}{3}}{s+1}$$

$$x(t) = \frac{1}{3}e^{2t} - \frac{1}{3}e^{-t}$$

Practice $x' - 4x = \cos t \quad , x(0) = 0$

2.4. Rules of Partial fractions

The table below shows all cases of rational function

Form of the rational function	Form of the partial fraction
$\frac{Px+q}{(x-a)(x-b)}, a \neq b$	$\frac{A}{x-a} + \frac{B}{x-b}$
$\frac{Px+q}{(x-a)^2}$	$\frac{A}{x-a} + \frac{B}{(x-a)^2}$
$\frac{px^2+qx+r}{(x-a)(x-b)(x-c)}$	$\frac{A}{x-a} + \frac{B}{x-b} + \frac{C}{x-c}$
$\frac{px^2+qx+r}{(x-a)^2(x-b)}$	$\frac{A}{x-a} + \frac{B}{(x-a)^2} + \frac{C}{x-b}$
$\frac{px^2+qx+r}{(x-a)(x^2+bx+c)}$	$\frac{A}{x-a} + \frac{Bx+C}{x^2+bx+c}$

Example 2.11

Solve the following differential equation by using Laplace transform.

$$Dx - x = 2\sin t \quad , x(0) = 0,$$

Solution

$$Dx - x = 2\sin t$$

$$\mathcal{L}[Dx] - \mathcal{L}[x] = \mathcal{L}[2\sin t]$$

$$s\mathcal{L}[x] - x(0) - \mathcal{L}[x] = \mathcal{L}[2\sin t]$$

$$s\mathcal{L}[x] - 0 - \mathcal{L}[x] = 2 * \frac{1}{s^2+1}$$

$$\mathcal{L}[x](s-1) = \frac{2}{s^2+1}$$

$$\mathcal{L}[x] = \frac{2}{(s^2+1)(s-1)}$$

By partial fractions

$$\mathcal{L}[x] = \frac{2}{(s^2+1)(s-1)} = \frac{A}{(s-1)} + \frac{Bs+C}{(s^2+1)}$$

$$2 = A\left(s^2+1\right) + Bs + C(s-1)$$

$$2 = As^2 + A + Bs^2 + Cs - Bs - C$$

$$2 = s^2(A+B) + s(C-B) + (A-C)$$

$$A + B = 0 \ \rightarrow A = -B \qquad C - B = 0 \ \rightarrow C = B$$

$$A - C = 2 \qquad A = 1, B = C = -1$$

$$\mathcal{L}[x] = \frac{1}{(s-1)} + \frac{-s-1}{(s^2+1)}$$

$$\mathcal{L}[x] = \frac{1}{(s-1)} - \frac{s}{(s^2+1)} - \frac{1}{(s^2+1)}$$

$$x(t) = \mathcal{L}^{-1}\left[\frac{1}{(s-1)}\right] - \mathcal{L}^{-1}\left[\frac{s}{(s^2+1)}\right] - \mathcal{L}^{-1}\left[\frac{1}{(s^2+1)}\right]$$

$$x(t) = e^t - cost - sint$$

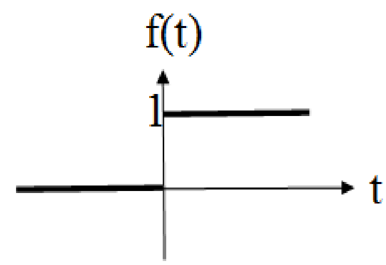
2.5 special cases

2.5.1 Heaviside unit step

Unit step functions are discontinuous function that means the function for a domain is different for another domain for example,

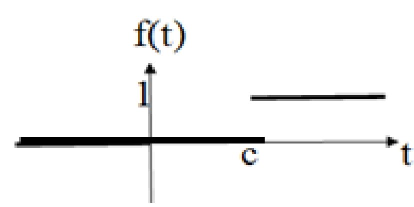
$$f(t) = \begin{cases} 0 & if\ t < 0 \\ 1 & if\ t \geq 0 \end{cases}$$

To represent this function graphically



To transform the step function by Laplace it to shift the function C and became

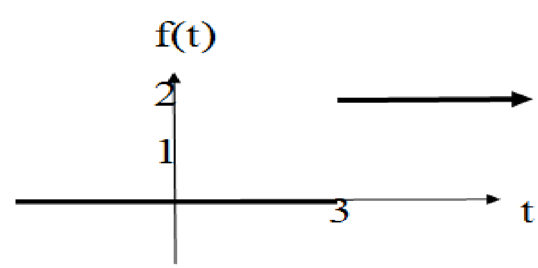
$$H(t-c) = \begin{cases} 0 & if\ t < c \\ 1 & if\ t \geq c \end{cases}$$



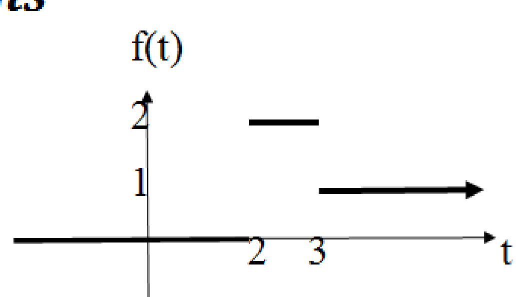
Some reference indicate H(t – c) as $u(t-c)$ **or** $u_c(t)$

For Example

$$f(t) = 2H(t-3)\ means$$



$$g(t) = 3H(t-2) - 2H(t-3)\ means$$

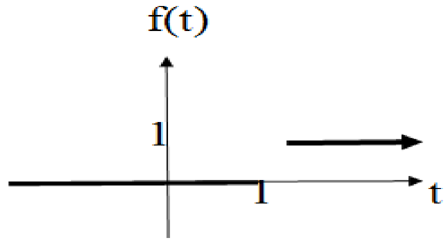


Example 2.12

Draw the function $f(t) = H(t-1) - H(t-3)$

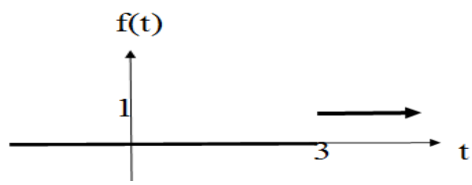
Solution

$H(t-1)$



$$H(t-1) = \begin{cases} 0 & if\ t < 1 \\ 1 & if\ t \geq 1 \end{cases}$$

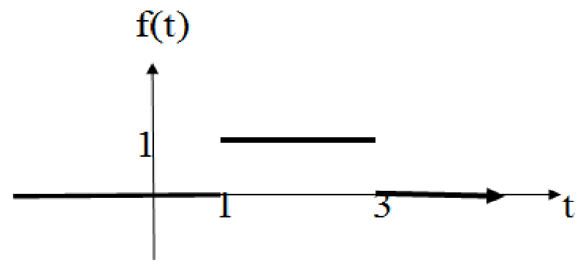
$H(t-3)$



$$H(t-3) = \begin{cases} 0 & if\ t < 3 \\ 1 & if\ t \geq 3 \end{cases}$$

$H(t-1) - H(t-3)$

$$H(t-1) - H(t-3) = \begin{cases} 0 & t < 1 \\ 1 & 1 \leq t < 3 \\ 0 & t \geq 3 \end{cases}$$



In general if the function

$H(t-a) - H(t-b)\ \ 0 \leq a < b$

$$H(t-a) - H(t-b) = \begin{cases} 0 & t < a \\ 1 & a \leq t < b \\ 0 & t \geq b \end{cases}$$

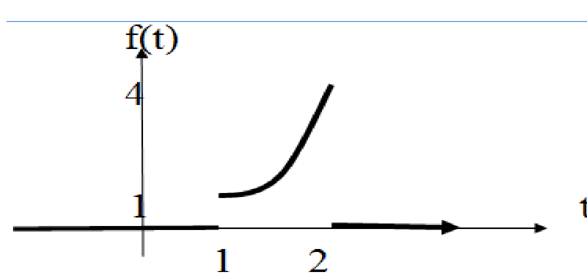
Notice that if the function

$$g(t)[H(t-a) - H(t-b)] = \begin{cases} 0 & t < a \\ g(t) & a \leq t < b \\ 0 & t \geq b \end{cases}$$

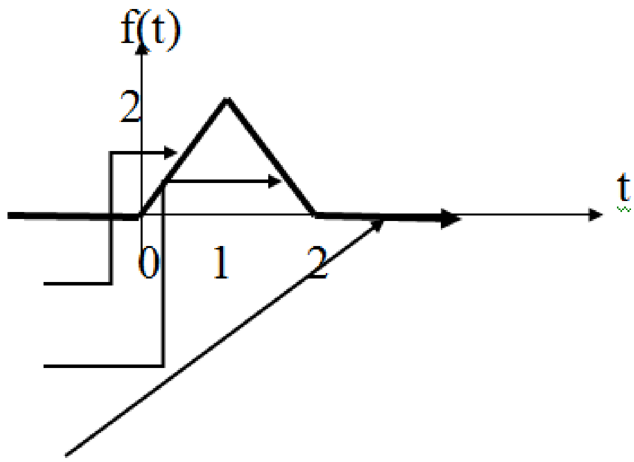
It is kind of integral of function between $[a, b]$

For example

$f(t) = t^2[H(t-1) - H(t-2)]$



Example 2.13



Solution

$$f(t) = 2t$$

$$f(t) = -2t + 4$$

$$f(t) = 0$$

$$f(t) = \begin{cases} 2t & 0 \le t < 1 \\ -2t + 4 & 1 \le t < 2 \\ 0 & t \ge 3 \end{cases}$$

$$f(t) = 2t[H(t-0) - H(t-1)] + (-2t+4)[H(t-1) - H(t-2)] + 0[H(t-2)]$$

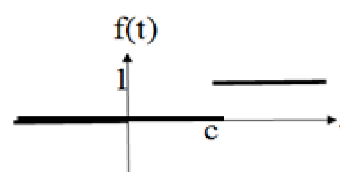
Approve

$$\mathcal{L}[f(t-c)H(t-c)] = e^{-cs}F(s) \quad ; s > 0$$

$$\mathcal{L}[f(t)] = F(s) = \int_0^{\infty} e^{-st} f(t)\,dt$$

$$\mathcal{L}[f(t-c)H(t-c)] = \int_0^{\infty} e^{-st} f(t-c)H(t-c)\,dt$$

$$\int_0^{c} e^{-st} f(t-c)H(t-c)\,dt + \int_c^{\infty} e^{-st} f(t-c)H(t-c)\,dt$$



$$= \int_c^{\infty} e^{-st} f(t-c)\,dt \quad assume\ \mathcal{T} = t - c\ \ and\ \mathcal{T} + c = t\ , d\mathcal{T} = dt$$

$$= \int_0^{\infty} e^{-s(\mathcal{T}+c)} f(\mathcal{T})\,d\mathcal{T} = \int_0^{\infty} e^{-s\mathcal{T}} e^{-sc} f(\mathcal{T})\,d\mathcal{T}$$

$$= e^{-sc} \int_0^{\infty} e^{-s\mathcal{T}} f(\mathcal{T})\,d\mathcal{T} = e^{-cs}F(s)$$

Example 2.13

Find $\mathcal{L}\left[t^2 H(t-1)\right]$

Solution

$$\mathcal{L}\left[t^2 H(t-1)\right] = \mathcal{L}[f(t-c)H(t-c)]$$

$$c = 1, \quad t^2 = f(t-c) \qquad let\ \mathcal{T} = t-1\ , \mathcal{T}+1 = t$$

$$(\mathcal{T}+1)^2 = f(\mathcal{T}) \qquad so\ \mathcal{L}\left[(\mathcal{T}+1)^2\right] = \mathcal{L}[f(\mathcal{T})]$$

$$\mathcal{L}[f(\mathcal{T})] = \mathcal{L}\left[\mathcal{T}^2 + 2\mathcal{T} + 1\right]$$

$$\mathcal{L}[1] = \frac{1}{s} \quad , \mathcal{L}[t^n] = \frac{n!}{s^{n+1}}$$

$$\mathcal{L}\left[\mathcal{T}^2\right] + 2\mathcal{L}[\mathcal{T}] + \mathcal{L}[1] = \frac{2}{s^3} + \frac{2}{s^2} + \frac{1}{s}$$

$$\mathcal{L}\left[t^2\ H(t-1)\right] = e^{-sc}\left(\frac{2}{s^2} + \frac{2}{s^2} + \frac{1}{s}\right)$$

Example 2.14

Find $\mathcal{L}[(e^t+1)H(t-2)]$

Solution

$$\mathcal{L}[(e^t+1)H(t-2)] = \mathcal{L}[f(t-2)H(t-2)]$$

$$c = 2, \quad e^t + 1 = f(t-2)$$

$$Let\ \mathcal{T} = t-2, \quad \mathcal{T}+2 = t$$

$$e^{\mathcal{T}+2} = f(t-2)$$

$$e^{\mathcal{T}+2} + 1 = f(\mathcal{T})$$

$$e^{\mathcal{T}}\ e^2 + 1 = f(\mathcal{T})$$

$$F(s) = \mathcal{L}[f(\mathcal{T})] = \mathcal{L}[e^2 * e^{\mathcal{L}}] + \mathcal{L}[1] = e^2\frac{1}{s-1} + \frac{1}{s}$$

$$\mathcal{L}[(e^t+1)H(t-2)] = e^{-2c}\left(e^2\frac{1}{s-1} + \frac{1}{s}\right)$$

Example 2.15

Solve $y^{//}+3y^{/}+2y=g(t)=\begin{cases}1 & 0\le t<10\\ 0 & t\ge 10\end{cases},\ y(0)=y(0)^{/}=0$

Solution

$$g(t)=H(t)-H(t-10)=1-H(t-10)$$

$$y^{//}+3y^{/}+2y=1-H(t-10)$$

Take Laplace for both sides

$$\mathcal{L}[y^{//}+3y^{/}+2y]=\mathcal{L}[1-H(t-10)]$$

$$\mathcal{L}[y^{//}]+3\mathcal{L}[y^{/}]+2\mathcal{L}[y]=\mathcal{L}[1]-\mathcal{L}[H(t-10)]$$

$$s^2\mathcal{L}[y(t)]+sy(0)+y(0)^{/}+3[\mathcal{L}[y(t)]+y(0)]+2\mathcal{L}[y(t)]=\mathcal{L}[1]-\mathcal{L}[H(t-10)]$$

$$\mathcal{L}[y(t)](s^2+3s+2)=\frac{1}{s}-\mathcal{L}[H(t-10)]$$

$$\mathcal{L}[1*H(t-10)]=\mathcal{L}[f(t-c)H(t-c)]=e^{-sc}F(s)$$

$$\mathcal{L}[1*H(t-10)]=e^{-10c}*\frac{1}{s}$$

$$\mathcal{L}[y(t)](s^2+3s+2)=\frac{1}{s}(1-e^{-10c})$$

$$\mathcal{L}[y(t)]=\frac{1}{s(s^2+3s+2)}(1-e^{-10c})=\frac{1}{s(s+2)(s+1)}(1-e^{-10c})$$

$$\frac{1}{s(s+2)(s+1)}=\frac{A}{s}+\frac{B}{(s+2)}+\frac{C}{(s+1)}$$

$$A(s+2)(s+1)+B\,s(s+2)+C\,s(s+1)=1$$

$$At\ s=0,-1,-2\ will\ get\ A=\frac{1}{2},B=\frac{1}{2},and\ C=-1$$

$$\mathcal{L}[y(t)]=\left[\frac{\frac{1}{2}}{s}+\frac{\frac{1}{2}}{(s+2)}-\frac{1}{(s+1)}(1-e^{-10s})\right]$$

$$y(t)=\mathcal{L}^{-1}\left[\frac{\frac{1}{2}}{s}+\frac{\frac{1}{2}}{(s+2)}-\frac{1}{(s+1)}\left(1-e^{-10s}\right)\right]$$

$$y(t)=\mathcal{L}^{-1}\left[\frac{\frac{1}{2}}{s}+\frac{\frac{1}{2}}{(s+2)}-\frac{1}{(s+1)}\right]-\mathcal{L}^{-1}\left[e^{-10s}\left(\frac{\frac{1}{2}}{s}+\frac{\frac{1}{2}}{(s+2)}-\frac{1}{(s+1)}\right)\right]$$

$$\frac{\frac{1}{2}}{s}+\frac{\frac{1}{2}}{(s+2)}-\frac{1}{(s+1)}=F(s)=\mathcal{L}[f(t)]$$

$$f(t)=\mathcal{L}^{-1}\left[\frac{\frac{1}{2}}{s}+\frac{\frac{1}{2}}{(s+2)}-\frac{1}{(s+1)}\right]=\frac{1}{2}+\frac{1}{2}e^{-2t}-e^{-t}$$

$$\left[e^{-10s}\left(\frac{\frac{1}{2}}{s}+\frac{\frac{1}{2}}{(s+2)}-\frac{1}{(s+1)}\right)\right]=e^{-sc}F(s)=\mathcal{L}[f(t-c)H(t-c]$$

$$f(t-10)=\frac{1}{2}+\frac{1}{2}e^{-2(t-10)}-e^{-(t-10)}$$

$$\mathcal{L}^{-1}\left[e^{-10s}\left(\frac{\frac{1}{2}}{s}+\frac{\frac{1}{2}}{(s+2)}-\frac{1}{(s+1)}\right)\right]=H(t-10)\left(\frac{1}{2}+\frac{1}{2}e^{-2(t-10)}-e^{-(t-10)}\right)$$

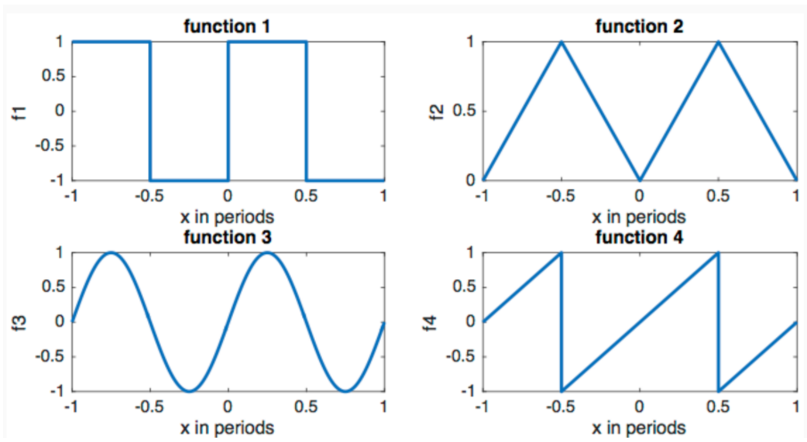
$$f(t)=\frac{1}{2}+\frac{1}{2}e^{-2t}-e^{-t}-H(t-10)\left(\frac{1}{2}+\frac{1}{2}e^{-2(t-10)}-e^{-(t-10)}\right)$$

Exercise

$$y^{//}+y=g(t)=\begin{cases}1 & 0\le t<3\pi \\ 0 & t\ge 3\pi\end{cases},\quad y(0)=0,\ y(0)^{/}=0$$

2.5.2 Periodic function

The periodic function is the function that repeats itself regularly as shown below



The period of full cycle on X-axis (t-axis) is donated by T as shown below.

$$g(t) = g(t+T) \;\; t > 0 \qquad g(t) \; has \; domain \; [0, \infty]$$

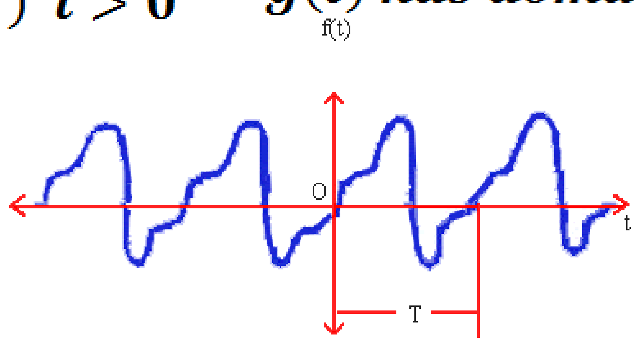


Fig. 7

Laplace of the periodic function is

$$\mathcal{L}[g(t)] = \frac{\int_0^T e^{-st} g(t)dt}{1 - e^{-sT}}$$

To proof this equation, we need to put the function in the Laplace form

Proof:

$$\mathcal{L}[g(t)] = \int_0^{\infty} e^{-st} g(t)dt$$

$$= \int_0^{T} e^{-st} g(t)dt + \int_T^{\infty} e^{-st} g(t)dt$$

$$Let\ \mathcal{T} = t - T \ , t = \mathcal{T} + T \ , d\mathcal{T} = dt$$

$$= \int_0^{T} e^{-st} g(t)dt + \int_0^{\infty} e^{-s(T+\mathcal{T})} g(T+\mathcal{T})d\mathcal{T}$$

$$= \int_0^{T} e^{-st} g(t)dt + \int_0^{\infty} e^{-sT} e^{-s\mathcal{T}} g(\mathcal{T})d\mathcal{T}$$

$$= \int_0^{T} e^{-st} g(t)dt + e^{-sT} \int_0^{\infty} e^{-s\mathcal{T}} g(\mathcal{T})d\mathcal{T}$$

$$\mathcal{L}[g(t)] = \int_0^{T} e^{-st} g(t)dt + e^{-sT} \mathcal{L}[g(t)]$$

$$\mathcal{L}[g(t)](1 - e^{-sT}) = \int_0^{T} e^{-st} g(t)dt$$

$$\mathcal{L}[g(t)] = \frac{\int_0^{T} e^{-st} g(t)dt}{1 - e^{-sT}}$$

Example 2.16

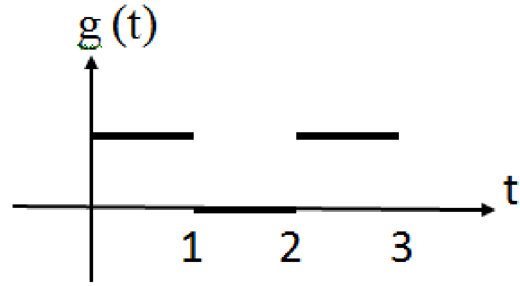
If g(t) has period 2 and g(t) is defined by:

$$g(t) = \begin{cases} 1 & if\ \ 0 \le t < 1 \\ 0 & if\ \ 1 \le t < 2 \end{cases}$$

Find $\mathcal{L}[g(t)]$ **?**

Solution ⟶

Solution



$$g(t) = g(t+2)$$

$$\mathcal{L}[g(t)] = \frac{\int_0^T e^{-st} g(t)dt}{1-e^{-sT}}$$

$$\mathcal{L}[g(t)] = \frac{\int_0^T e^{-st} g(t)dt}{1-e^{-sT}} = \frac{\int_0^2 e^{-st} g(t)dt}{1-e^{-2s}}$$

$$= \frac{\int_0^1 e^{-st} * 1dt + \int_1^2 e^{-st} * 0dt}{1-e^{-2s}} = \frac{\int_0^1 e^{-st} dt}{1-e^{-2s}}$$

$$= \frac{1}{1-e^{-2s}} \left[\frac{e^{-st}}{-s}\right]_0^1 = \frac{1}{1-e^{-2s}} \left[\frac{e^{-s}}{-s} + \frac{1}{s}\right]$$

$$= \frac{1}{1-e^{-2s}} \left[\frac{1-e^{-s}}{s}\right] = \frac{1}{(1-e^{-s})(1+e^{-s})} \left[\frac{1-e^{-s}}{s}\right]$$

$$\mathcal{L}[g(t)] = \frac{1}{s+se^{-s}}$$

Practice Find

$$\mathcal{L}[g(t)] \; if \; g(t) = \begin{cases} 0 & if \;\; 0 \le t < 2 \\ 1 & if \;\; 2 \le t < 4 \end{cases}$$

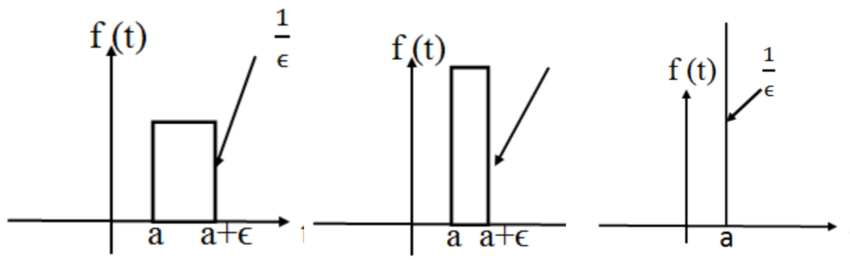
5.2.3 Dirac Delta function

Dirac's delta function is defined by the following property.

$$\delta(t) = \begin{cases} 0 & t \neq 0 \\ \infty & t = 0 \end{cases}$$

$$\int_{t1}^{t2} \delta(t)dt = 1$$

$$0 \in [t_1, t_2]$$



$\epsilon \to 0, \delta(t) = \frac{1}{\epsilon} \to \infty$ at a

It is "infinitely peaked" at t = 0 with the total area of unity.

The important property of the delta function is the following relation.

$$\int \delta(t)f(t)dt = f(0)$$

For any function f(t). This is easy to see. First of all, δ(t) vanishes everywhere except t = 0. Therefore, it does not matter what values the function f(t) takes except at t = 0. You can then say f(t)δ(t) = f(0) δ(t). Then f(0) can be pulled outside the integral because it does not depend on t, and you obtain the r.h.s. This equation can easily be generalized to

$$\int \delta(t)f(t - t_0)dt = f(t_0)$$

For examples

$$\int_{-5}^{5} 7e^{t^2}\cos(t)\,\delta(t)\,dt = 7 \qquad ,at\ t=0$$

$$\int_{-5}^{5} 7e^{t^2}\cos(t)\,\delta(t-2)\,dt = 7e^{4}\cos(2) \qquad ,at\ t=2$$

$$\int_{-5}^{1} 7e^{t^2}\cos(t)\,\delta(t-2)\,dt = 0 \qquad ,at\ t=2$$

Example 2.17

Solve $x^{//} + x = \delta(t-\pi)\ , x(0) = x(0)^{/} = 0$

Solution

$$s^2\mathcal{L}[x(t)] + sx(0) + x(0)^{/} + \mathcal{L}[x(t)] = e^{-\pi s}$$

$$\mathcal{L}[x(t)]\left(s^2+1\right) = e^{-\pi s}$$

$$\mathcal{L}[x(t)] = \frac{e^{-\pi s}}{(s^2+1)}$$

$$x(t) = \mathcal{L}^{-1}\left[\frac{e^{-\pi s}}{(s^2+1)}\right] = f(t-\pi) * H(t-\pi)$$

Since $\mathcal{L}[f(t-c) * H(t-c)] = e^{-sc}F(s)$

It can be compared

$$\mathcal{L}^{-1}\left[\frac{e^{-\pi s}}{(s^2+1)}\right] = f(t-\pi) * H(t-\pi)$$

$$\mathcal{L}^{-1}F(s) = \mathcal{L}^{-1}\frac{1}{s^2+1} = sint = f(t)$$

$$x(t) = sin(t-\pi) * H(t-\pi)$$

Example 2.18

Solve $x^{//} + x = f(t)$

Where f(t) is a hummer hit the system at any $t = n\pi \;\; n > 0$

Solution

$$\mathcal{L}[x(t)](s^2+1) = \sum_{n=0}^{\infty} \mathcal{L}\,[\delta(t-n\pi)]$$

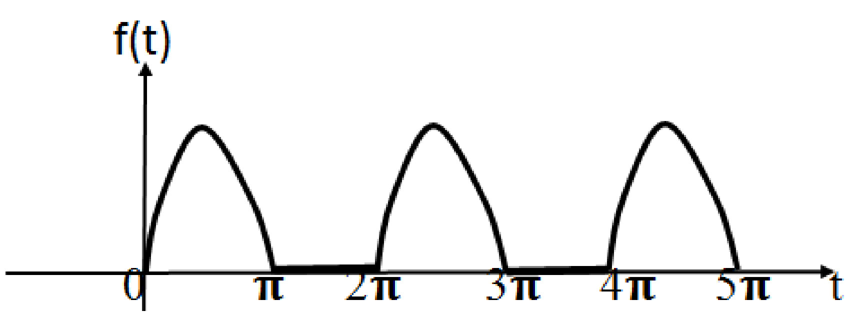
$$\mathcal{L}[x(t)] = \sum_{n=0}^{\infty} \frac{e^{-n\pi s}}{(s^2+1)}$$

$$x(t) = \sum_{n=0}^{\infty} H(t-n\pi)\sin(t-n\pi)$$

$$For\; n\pi < t < (n+1)\pi$$

$$x(t) = sint - sint + \cdots + (-1)^n sint$$

$$x(t) = \begin{cases} sint & if\;\; n\, is\, even \\ 0 & if\;\; n\, is\, odd \end{cases}$$



5.2.4 Convolution theorem

The general form of convolution

$$f(t) * g(t) = \int_0^t f(\mathcal{T})g(t-\mathcal{T})d\mathcal{T}$$

5.2.4.1 Properties

1. $f * g = g * f$
2. $f * (g * h) = (f * g) * h$
3. $f * (g + h) = (f * g) + (f * h)$

Laplace transform

$$\mathcal{L}[f(t) + g(t)] = \mathcal{L}[f(t)] + \mathcal{L}[g(t)] = F(s) + G(s)$$

$$\mathcal{L}[f(t)\ g(t)] \neq \mathcal{L}[f(t)]\mathcal{L}[g(t)]$$

But

$$\mathcal{L}[f(t) * g(t)] = \mathcal{L}[f(t)]\mathcal{L}[g(t)]$$

Convolution Multiplication

$$\mathcal{L}^{-1}[F(s)G(s)] = f(t) * g(t) = \int_0^t f(\mathcal{T})g(t-\mathcal{T})d\mathcal{T}$$

Example 2.19

Find $\mathcal{L}^{-1}\left[\frac{1}{(s^2+1)^2}\right]$

Solution

$$\mathcal{L}^{-1}\left[\frac{1}{(s^2+1)^2}\right] = \mathcal{L}^{-1}\left[\frac{1}{(s^2+1)(s^2+1)}\right]$$

Since

$$\mathcal{L}^{-1}\left[\frac{1}{(s^2+1)}\right] = sint$$

$$\mathcal{L}^{-1}\left[\frac{1}{(s^2+1)(s^2+1)}\right] = sint * sint$$

$$= \int_0^t sin(\mathcal{T})\, sin(t-\mathcal{T})d\mathcal{T}$$

Remember

$$\sin a \sin b = \frac{1}{2}[cos(b-a) - \cos(b+a)]$$

$$= \frac{1}{2}\int_0^t [cos(t-2\mathcal{T}) - \cos(t)]d\mathcal{T}$$

$$= \frac{1}{2}\left[\frac{sin(2\mathcal{T}-t)}{2}\right]_0^t - \frac{1}{2}tcost$$

$$= \frac{1}{2}\left[\frac{sint}{2} - \frac{sin(-t)}{2}\right] - \frac{1}{2}tcost$$

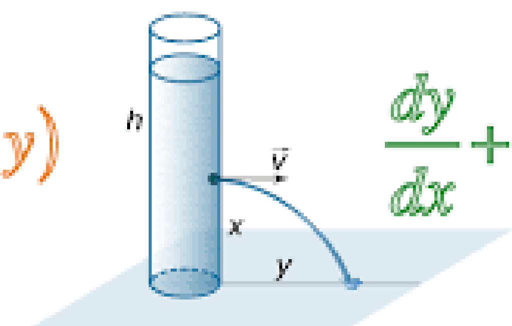
$$= \frac{1}{2}sint - \frac{1}{2}tcost$$

Practice: solve $\frac{dy}{dt} - ay = e^{ct}$

Chapter three

System of Linear differential equations

$$\frac{dy}{dx} = f(x, y)$$



$$\frac{dy}{dx} + a(x)y = f(x)$$

3.1 Definition

A system of ordinary differential equations is two or more equations involving the derivatives of two or more unknown functions of a single independent variable.

Example:

$$\frac{dx}{dt} = f(x,y,t) \; and \; \frac{dy}{dt} = f(x,y,t)$$

Where t is an independent variable and (x,y) are dependent variables.

A solution of a system, such as above, is a pair of differentiable functions x = φ_1(t) and y = φ_2(t) defined on a common interval I that satisfy each equation of the system on this interval.

3.2 Solving linear system (Elimination method)

The elimination method consists in bringing the system of nth differential equations into a single differential equation of order n. The following example explains this.

Example 3.1

Solve the following system of differential equations

$$\frac{dx}{dt} = y \qquad \frac{dy}{dt} = 3x$$

Solution

$$\frac{d}{dt} = D(Operater)$$
$$Dx = y$$
$$Dy = 3x$$

To eliminate (y)

$$Dx = y \qquad (multiply\ by\ D)$$
$$Dy = 3x$$

Became

$$D^2x - Dy = 0$$
$$-3x + Dy = 0$$

By adding the above two equations

$$D^2x - 3x = 0$$

$(D^2 - 3)x = 0$ **the characteristic equation is**

$$m^2 - 3 = 0 \rightarrow m_1 = \sqrt{3}\ and\ m_2 = -\sqrt{3}$$

$$x(t) = C_1 e^{\sqrt{3}t} + C_2 e^{-\sqrt{3}t}$$

To eliminate (x) recall the main equations

$$Dx - y = 0 \quad (multiply\ by\ 3)$$
$$-3x + Dy = 0 \quad (multiply\ by\ D)$$

They became

$$3Dx - 3y = 0$$

$$-3Dx + D^2y = 0$$

By adding the above two differential equations and became

$(D^2 - 3)y = 0$ **the characteristic equation is**

$$m^2 - 3 = 0 \rightarrow m_1 = \sqrt{3} \ and \ m_2 = -\sqrt{3}$$

$$y(t) = C_3e^{\sqrt{3}t} + C_4e^{-\sqrt{3}t}$$

To find the relations between Cs

From the main equation

$$\frac{dx}{dt} = y$$

$$\sqrt{3}C_1e^{\sqrt{3}t} - \sqrt{3}C_2e^{-\sqrt{3}t} = C_3e^{\sqrt{3}t} + C_4e^{-\sqrt{3}t}$$

$$C_3 = \sqrt{3}C_1 \ and \ C_4 = -\sqrt{3}C_2$$

The final results

$$x(t) = C_1e^{\sqrt{3}t} + C_2e^{-\sqrt{3}t}$$

$$y(t) = \sqrt{3}C_1e^{\sqrt{3}t} - \sqrt{3}C_2e^{-\sqrt{3}t}$$

Practices

Solve the systems of linear differential equations

$(a) \quad \frac{dx}{dt} = 4x + 7y \quad , \frac{dy}{dt} = x - 2y$

$(b) \ \frac{dx}{dt} + \frac{dy}{dt} = e^t \quad , - \frac{d2x}{dt^2} + \frac{dx}{dt} = -x - y$

3.3 Applications of system linear algebra

A linear equation in the variables $x_1,\ldots,x_n$ is an equation that can be written in the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$$

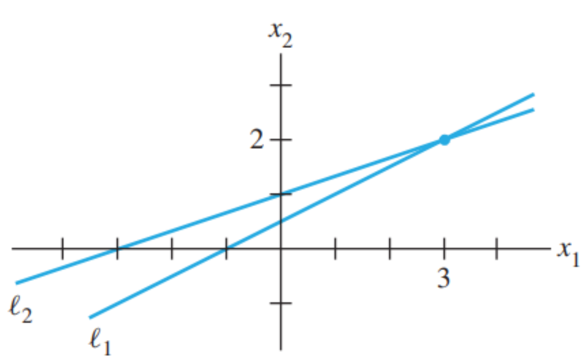
where b and the coefficients $a_1,\ldots,a_n$ are real or complex numbers, usually known in advance. The subscript n may be any positive integer.

A solution of the system is a list .$s_1, s_2,\ldots,s_n$ of numbers that makes each equation a true statement when the values .$s_1,\ldots,s_n$ are substituted for $x_{1,\ldots,}x_n$, respectively

For example

$$\begin{aligned} x_1 - 2x_2 &= -1 \\ -x_1 + 3x_2 &= 3 \end{aligned}$$

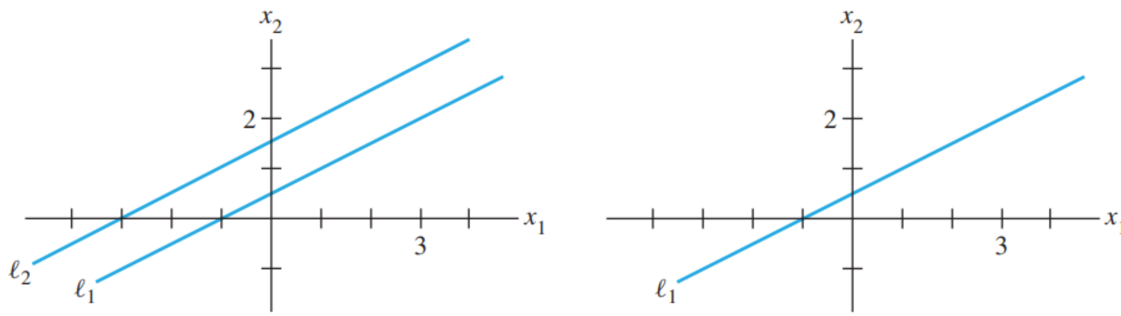
The solution of this system is to find x_1 and x_2 those satisfy the both above equations, in other word, the intersection point is the solution as shown in the figure



Sometime the system does not have a solution as shown below

(a) $$\begin{aligned} x_1 - 2x_2 &= -1 \\ -x_1 + 2x_2 &= 3 \end{aligned}$$

(b) $$\begin{aligned} x_1 - 2x_2 &= -1 \\ -x_1 + 2x_2 &= 1 \end{aligned}$$



(a) (b)

Example 3.2

Solve the linear system of equations

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ 2x_2 - 8x_3 &= 8 \\ 5x_1 - 5x_3 &= 10 \end{aligned}$$

Solution

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ 2x_2 - 8x_3 &= 8 \\ 5x_1 \qquad - 5x_3 &= 10 \end{aligned} \qquad \left|\begin{array}{cccc} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 5 & 0 & 5 & 10 \end{array}\right|$$

$$(Equation\ 3) R_3 \to -5R_1 + R_3$$

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ 2x_2 - 8x_3 &= 8 \\ 10x_2 - 10x_3 &= 10 \end{aligned} \qquad \left|\begin{array}{cccc} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 0 & 10 & -10 & 10 \end{array}\right|$$

$(Equation\ 2) R_2/2$

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ x_2 - 4x_3 &= 4 \\ 10x_1 - 10x_3 &= 10 \end{aligned} \qquad \begin{vmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 10 & -10 & 10 \end{vmatrix}$$

$(Equation\ 3) R_3 \rightarrow -10R_2 + R_3$

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ x_2 - 4x_3 &= 4 \\ 30x_3 &= -30 \end{aligned} \qquad \begin{vmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 30 & -30 \end{vmatrix}$$

$(Equation\ 3) R_3 \rightarrow \frac{1}{30} R_3$

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ x_2 - 4x_3 &= 4 \\ x_3 &= -1 \end{aligned} \qquad \begin{vmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 1 & -1 \end{vmatrix}$$

$x_3 = -1$

$x_2 - 4(-1) = 4 \rightarrow x_2 = 0$

$x_1 - 2(0) + 1(-1) = 0 \rightarrow x_1 = 1$

Practice

Solve the following system linear equations

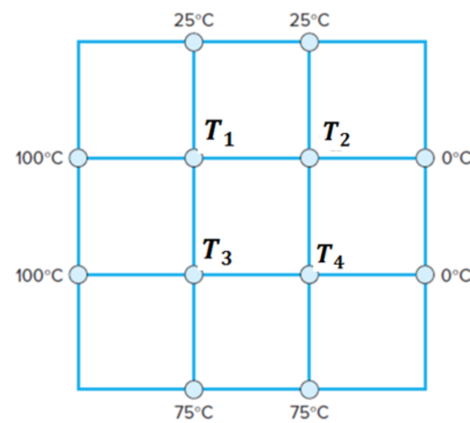
$$\begin{aligned} x_1 + 4x_2 - 2x_3 + 8x_4 &= 12 \\ x_2 - 7x_3 + 2x_4 &= -4 \\ 5x_3 - x_4 &= 7 \\ x_3 + 3x_4 &= -5 \end{aligned}$$

Temperature distribution can be an application of system of linear equations

Example 3.3

A steady state temperature distribution of the plate shown below, find the temperature distribution $T_1, T_2, T_3 \, and \, T_4$

Solution



The heat balance means (steady state)

$T_1, T_2, T_3 \, and \, T_4$ **are became constants.**

$$T_1 = \frac{1}{4}[100 + 25 + T_2 + T_3]$$

$$T_2 = \frac{1}{4}[0 + 25 + T_1 + T_4]$$

$$T_3 = \frac{1}{4}[100 + 75 + T_1 + T_4]$$

$$T_4 = \frac{1}{4}[0 + 75 + T_2 + T_3]$$

The equation can be arranged as

$$4T_1 - T_2 - T_3 + 0T_4 = 125$$

$$-T_1 + 4T_2 + 0T_3 - T_4 = 25$$

$$-T_1 + 0T_2 + 4T_3 - T_4 = 175$$

$$0T_1 - T_2 - T_3 + 4T_4 = 75$$

$$\left|\begin{matrix} 4 & -1 & -1 & 0 & 125 \\ -1 & 4 & 0 & -1 & 25 \\ -1 & 0 & 4 & -1 & 175 \\ 0 & -1 & -1 & 4 & 75 \end{matrix}\right|$$

$$(Equ.2) R_2 \to \frac{1}{4}R_1 + R_2$$

$$(Equ.3) R_3 \to \frac{1}{4}R_1 + R_3$$

The system became

$$4T_1 - T_2 - T_3 + 0T_4 = 125$$
$$0T_1 + \frac{15}{4}T_2 - \frac{1}{4}T_3 - T_4 = \frac{225}{4}$$
$$0T_1 - \frac{1}{4}T_2 + \frac{15}{4}T_3 - T_4 = \frac{825}{4}$$
$$0T_1 - T_2 - T_3 + 4T_4 = 75$$

$$\left|\begin{matrix} 4 & -1 & -1 & 0 & 125 \\ 0 & \frac{15}{4} & \frac{-1}{4} & -1 & \frac{225}{4} \\ 0 & \frac{-1}{4} & \frac{15}{4} & -1 & \frac{825}{4} \\ 0 & -1 & -1 & 4 & 75 \end{matrix}\right|$$

$$(Equ.3)R_3 \to \frac{1}{15}R_2 + R_3 \qquad (Equ.4)R_4 \to \frac{4}{15}R_2 + R_4$$

$$4T_1 - T_2 - T_3 + 0T_4 = 125$$
$$0T_1 + \frac{15}{4}T_2 - \frac{1}{4}T_3 - T_4 = \frac{225}{4}$$
$$0T_1 + 0T_2 + \frac{224}{60}T_3 - \frac{16}{15}T_4 = 210$$
$$0T_1 + 0T_2 - \frac{16}{15}T_3 + \frac{56}{15}T_4 = 90$$

$$\left|\begin{matrix} 4 & -1 & -1 & 0 & 125 \\ 0 & \frac{15}{4} & \frac{-1}{4} & -1 & \frac{225}{4} \\ 0 & 0 & \frac{224}{60} & -\frac{16}{15} & 210 \\ 0 & 0 & \frac{-16}{15} & \frac{56}{15} & 90 \end{matrix}\right|$$

$$(Equ.4)R_4 \to \frac{2}{7}R_3 + R_4$$

$$4T_1 - T_2 - T_3 + 0T_4 = 125$$
$$0T_1 + \frac{15}{4}T_2 - \frac{1}{4}T_3 - T_4 = \frac{225}{4}$$
$$0T_1 + 0T_2 + \frac{224}{60}T_3 - \frac{16}{15}T_4 = 210$$
$$0T_1 + 0T_2 + 0T_3 + 3.428T_4 = 150$$

$$\left|\begin{matrix} 4 & -1 & -1 & 0 & 125 \\ 0 & \frac{15}{4} & \frac{-1}{4} & -1 & \frac{225}{4} \\ 0 & 0 & \frac{224}{60} & -\frac{16}{15} & 210 \\ 0 & 0 & 0 & 3.428 & 150 \end{matrix}\right|$$

$$.428T_4 = 150 \to T_4 = \frac{150}{3.428} = 43.75C^o$$

$$\frac{224}{60}T_3 - \left(\frac{16}{15}\right)(43.75) = 210 \rightarrow T_3 = 68.75C^o$$

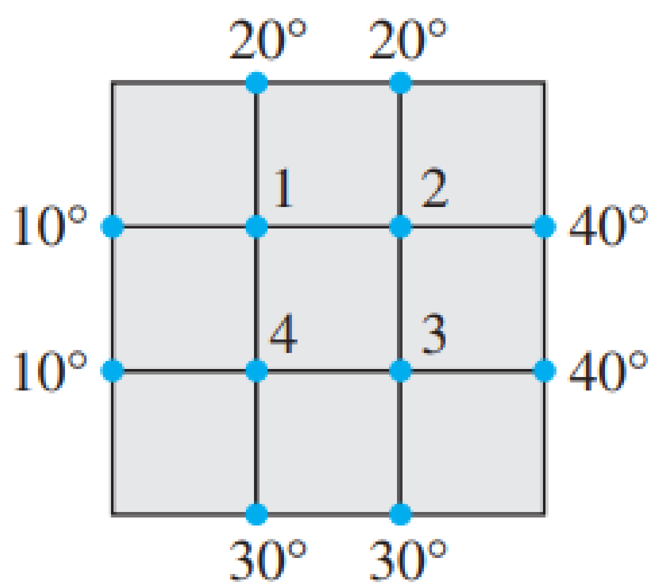
$$\frac{15}{4}T_2 - \left(\frac{1}{4}\right)(68.75) - 43.75 = \frac{225}{4} \rightarrow T_2 = 31.25C^o$$

$$4T_1 - 31.25 - 68.754 = 125 \rightarrow T_1 = 56.25C^o$$

$$\left(T_1 = 56.25C^o, T_2 = 31.25C^o, T_3 = 68.75C^o, T_4 = 43.75C^o,\right)$$

Practice

Find the temperature distribution of the plate below at steady state case.



3.4 Homogenous linear system differential equations

If $\frac{dx}{dt} = x + 2y \qquad \frac{dy}{dt} = 3x + 2y$

To write it by matrix form

$$\begin{pmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\downarrow \qquad \downarrow \qquad \downarrow$$

$$\bar{X} = \quad A \qquad X$$

To solve the system we get

$$x = 2e^{4t} \quad and \quad y = 3e^{4t}$$

$$x = 2e^{4t}, \qquad \frac{dx}{dt} = 8e^{4t}$$

$$y = 3e^{4t}, \qquad \frac{dy}{dt} = 12e^{4t}$$

They can be substituted in the main equations

$$8e^{4t} = (2e^{4t}) + 2(3e^{4t}) \rightarrow 8e^{4t} = 8e^{4t}$$

$$12e^{4t} = 3(2e^{4t}) + 2(3e^{4t}) \rightarrow 12e^{4t} = 12e^{4t}$$

Example 3.4

Show that $x = e^{2t}, y = 2e^{2t} \quad (1)$

$$x = e^{3t}, \qquad y = e^{3t} \quad (2)$$

Are the solutions of the following system

$$\frac{dx}{dt} = 4x - y \qquad \frac{dy}{dt} = 2x + y$$

Solution

<u>The first one</u>

$$2e^{2t} = 4(e^{2t}) - 2(e^{2t}) \to 2e^{2t} = 2e^{2t}$$

$$4e^{2t} = 2(e^{2t}) + 2(e^{2t}) \to 4e^{2t} = 4e^{2t}$$

<u>The second one</u>

$$3e^{3t} = 4(e^{3t}) - (e^{3t}) \to 3e^{3t} = 3e^{3t}$$

$$3e^{3t} = 2(e^{3t}) + (e^{3t}) \to 3e^{3t} = 3e^{3t}$$

The homogenous linear system can be solved by eigenvalues and eigenvectors method.

In general

$$\frac{dx}{dt} = ax + by \qquad \frac{dy}{dt} = cx + dy$$

Or

$$\bar{X} = A \ X$$

The solution is

$$x = \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} e^{\lambda t}$$

λ is called <u>Lamda</u>

For system with two dependent variables

$$x_1 = \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} e^{\lambda_1 t} \qquad x_2 = \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} e^{\lambda_2 t}$$

The solution equivalent is

$$X = K \; e^{\lambda t}$$

$$\bar{X} = \lambda K \; e^{\lambda t}$$

From the system $\bar{X} = A \; X$

Then

$$\lambda K \; e^{\lambda t} = AKe^{\lambda t}$$

$$\lambda K \;\; = AK$$

$$\rightarrow \; AK - \lambda K = 0 \qquad \rightarrow K(A - \lambda I) \; = 0$$

Two possibilities

$$K = 0 \,, or \;\; (A - \lambda I) = 0$$

$$Det. \; (A - \lambda I) = 0$$

- **The solution for λ in the characteristic equation (eigenvalues)**
- **The vectors corresponding with each value of λ called eigenvectors**

Example 3.5

Solve linear system using Eigen value and Eigenvectors for the homogenous differential equations

$$\frac{dx}{dt} = 4x - y \qquad \frac{dy}{dt} = 2x + y$$

Solution

$$\bar{X} = A\,X \qquad A = \begin{pmatrix} 4 & -1 \\ 2 & 1 \end{pmatrix}$$

$$Det.\ (A - \lambda I) = 0$$

$$Det.\ \begin{pmatrix} 4-\lambda & -1 \\ 2 & 1-\lambda \end{pmatrix} = (4-\lambda)(1-\lambda) - (2)(-1) = 0$$

$$\lambda^2 - 5\lambda + 6 = 0$$

$$(\lambda - 2)(\lambda - 3) = 0 \rightarrow \lambda_1 = 2, \lambda_2 = 3$$

$$\boxed{\lambda_1 = 2}$$

$$(A - \lambda I) = 0$$

$$\begin{pmatrix} 4-2 & -1 \\ 2 & 1-2 \end{pmatrix} = \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} = 0 \rightarrow \begin{pmatrix} 2 & -1 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} k_1 \\ k_2 \end{pmatrix} = 0$$

$$2k_1 - k_2 = 0$$

$$2k_1 - k_2 = 0$$

$2\,k_1 = k_2 \rightarrow at\ k_1 = 1\,, k_2 = 2$ **Eigenvalues**

$$x = \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{2t}$$

$$\boxed{\lambda_2 = 3}$$

$$(A - \lambda I) = 0$$

$$\begin{pmatrix} 4-3 & -1 \\ 2 & 1-3 \end{pmatrix} = \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} = 0 \rightarrow \begin{pmatrix} 1 & -1 \\ 2 & -2 \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} = 0$$

$$k_1 - k_2 = 0$$

$$2k_1 - 2k_2 = 0$$

$$k_1 = k_2 \rightarrow at\ k_1 = 1, k_2 = 1$$

$$x = \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{3t}$$

$$X = c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{3t}$$

Example 3.6

Solve linear system using Eigen value and Eigenvectors for the homogenous differential equations.

$$\frac{dx}{dt} = 2y \qquad \frac{dy}{dt} = -6x + 7y$$

Solution

$$\bar{X} = A\,X$$

$$A = \begin{pmatrix} 0 & 2 \\ -6 & 7 \end{pmatrix}$$

$$Det.(A - \lambda I) = 0 \quad \rightarrow = \begin{pmatrix} -\lambda & 2 \\ -6 & 7-\lambda \end{pmatrix} = 0$$

$$(-\lambda)(7-\lambda) - (2)(-6) = 0 \rightarrow -7\lambda - \lambda^2 + 12 = 0$$

$$\lambda^2 - 7\lambda + 12 = 0 \rightarrow (\lambda - 3)(\lambda - 4) = 0$$

$$\lambda_1 = 3, \lambda_1 = 4 \ \ eigenvalues$$

$\boxed{\lambda_1 = 3}$

$$(A-\lambda I)K = 0 \rightarrow \begin{pmatrix} -3 & 2 \\ -6 & 4 \end{pmatrix}\begin{pmatrix} k_1 \\ k_2 \end{pmatrix} = 0$$

$$-3k_1 + 2k_2 = 0$$

$$-6k_1 + 4k_2 = 0$$

$3k_1 = 2k_2 \rightarrow at\ k_1 = 2\ , k_2 = 3$ *Eigenvectors* $k_1 = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$

$\boxed{\lambda_1 = 4}$

$$(A-\lambda I)K = 0 \rightarrow \begin{pmatrix} -4 & 2 \\ -6 & 3 \end{pmatrix}\begin{pmatrix} k_1 \\ k_2 \end{pmatrix} = 0$$

$$-4k_1 + 2k_2 = 0$$
$$-6k_1 + 3k_2 = 0$$

$2k_1 = k_2 \rightarrow at\ k_1 = 1\ , k_2 = 2$ *Eigenvectors* $k_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

$$X = C_1\begin{pmatrix} 2 \\ 3 \end{pmatrix}e^{3t} + C_2\begin{pmatrix} 1 \\ 2 \end{pmatrix}e^{4t}$$

•

solving systems by Laplace transforms

In additional to solve single differential equation, Laplace transform method can solve system of differential equations.

Example 3.7

Solve the system of differential equations by Laplace transform.

$$2x' + y' - y = t, \qquad x(0) = 1$$
$$x' + y' = t^2, \qquad y(0) = 0$$

Solution

Take Laplace transform for both equations

$$2\left[s\mathcal{L}[x(t) - x(0)]\right] + s\mathcal{L}[y(t) - y(0)] - \mathcal{L}[y(0)] = \frac{1}{s^2}$$

$$s\mathcal{L}[x(t) - x(0)] + s\mathcal{L}[y(t) - y(0)] = \frac{2}{s^3}$$

Then

$$2s\mathcal{L}[x(t)] + (s-1)\mathcal{L}[y(t)] = 2 + \frac{1}{s^2}$$

$$2\left(s\mathcal{L}[x(t)] + s\mathcal{L}[y(t)] = 1 + \frac{2}{s^3}\right) \qquad (Multiply\ by\ 2)$$

By subtraction the two equation

$$(s - 1 - 2s)\mathcal{L}[y(t)] = \frac{1}{s^2} - \frac{4}{s^3}$$

$$(-s-1)\mathcal{L}[y(t)] = \frac{s-4}{s^3}$$

$$\mathcal{L}[y(t)] = \frac{4-s}{(s+1)s^3}$$

By partial fraction

$$\frac{4-s}{(s+1)s^3} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s^3} + \frac{D}{(s+1)}$$

$$4 = A\,s^2(s+1) + Bs(s+1) + C(s+1) + Ds^3$$

$$A + D = 0$$

$$A + B = 0$$

$$B + C = -1$$

$$C = 4 \rightarrow B = -5, A = 5 \; and \; D = -5$$

$$\mathcal{L}[y(t)] = \frac{5}{s} - \frac{5}{s^2} + \frac{4}{s^3} - \frac{5}{s+1}$$

$$y(t) = \mathcal{L}^{-1}\left[\frac{5}{s} - \frac{5}{s^2} + \frac{4}{s^3} - \frac{5}{s+1}\right]$$

$$y(t) = 5 - 5t + 2t^2 - 5e^{-t}$$

Recall the equation

$$s\mathcal{L}[x(t)] - 1 + s\mathcal{L}[y(t)] = \frac{2}{s^3} \qquad (divide\ by\ s)$$

$$\mathcal{L}[x(t)] = \frac{1}{s} + \frac{2}{s^3} - \mathcal{L}[y(t)]$$

$$\mathcal{L}[x(t)] = \frac{1}{s} + \frac{2}{s^3} - \left[\frac{5}{s} - \frac{5}{s^2} + \frac{4}{s^3} - \frac{5}{s+1}\right]$$

$$x(t) = 1 + \frac{2*1}{6}t^3 - \left[5 - 5t + 2t^2 - 5e^{-t}\right]$$

$$x(t) = -4 + 5t - 2t^2 + \frac{1}{3}t^3 + 5e^{-t}$$

Sheet 3

1- Solve the following systems linear differential equations (by elimination)

$$(a)\ \frac{dx}{dt} = -9y\,, \qquad \frac{dy}{dt} = -4x$$

$$(b)\ \frac{dx}{dt} = x - y + e^{t}\,, \qquad \frac{dy}{dt} = x + 3y$$

$$(c)\ \frac{dx}{dt} = y - t\,, \qquad \frac{dy}{dt} = x + t$$

2- Solve the following systems linear following homogenous differential equations (by eigenvalues and eigenvectors methods)

$$(a)\ \frac{dx}{dt} = 5x + 2y, \qquad \frac{dy}{dt} = 2x + +5y$$

$$(b)\ \overline{X} = \begin{vmatrix} 1 & -5 \\ 1 & -1 \end{vmatrix} X$$

3- Solve the following systems linear differential equations (by using Laplace transform)

$$(a)\ \frac{dx}{dt} + y = e^{-t}, \qquad \frac{dy}{dt} - x = 3e^{-t}\,, x(0) = 0, y(0) = 1$$

$$(b)\ \frac{dx}{dt} + y = 1, \qquad \frac{dy}{dt} - x = 0\,, x(0) = -1\,, y(0) = 1$$

Chapter Four

Partial Differential equations

$$\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2} + f(t, x)$$

4.1 What is the Partial differential equations

The key defining property of a partial differential equation (PDE) is that there is more than one independent variable x, y,.... There is a dependent variable that is an unknown function of these variables u(x, y, . . .). We will often denote its derivatives by subscripts; thus ∂u/∂x = ux , and so on. A PDE is an identity that relates the independent variables, the dependent variable u, and the partial derivatives of u. It can be written as

The general Form

$$F\left(x_1, x_2, x_3, \dots\dots, x_n, \frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, \frac{\partial u}{\partial x_3}, \frac{\partial 2u}{\partial x_1 \partial x_2}, \dots\right) = 0$$

Notation that

$$u_x = \frac{\partial u}{\partial x} \ and \ u_{xy} = \frac{\partial 2u}{\partial x \partial y} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right)$$

4.2 Classification of PDEs (Linear or Nonlinear

4.2.1 Linear equation

The general form

$$a_1(x,y)u_x + a_2(x,y)u_y = 0$$ **(homogenous)**

$$a_1(x,y)u_x + a_2(x,y)u_y = f(x,y)$$ **(non homogenous)**

4.2.2 Quasilinear equation (Non-Linear)

The general form

$$a_1(x,y,u)u_x + a_2(x,y,u)u_y = 0$$

(homogenous)

$$a_1(x,y,u)u_x + a_2(x,y,u)u_y = f(x,y)$$

(non homogenous)

4.3 Some famous equations (Linear & 2nd Order)

$$1- \ \frac{\partial 2u}{\partial t^2} = C^2 \frac{\partial 2u}{\partial x^2} \quad (1D\ wave\ equation)$$

$$2- \ \frac{\partial u}{\partial t} = C^2 \frac{\partial 2u}{\partial x^2} \quad (1D\ Heat\ equation)$$

$$3- \ \frac{\partial 2u}{\partial x^2} + \frac{\partial 2u}{\partial y^2} = 0 \quad (2D\ Laplace\ equation)$$

$$4- \ \frac{\partial 2u}{\partial x^2} + \frac{\partial 2u}{\partial y^2} = f(x,y) \quad (2D\ poison\ equation)$$

$$5- \ \frac{\partial 2u}{\partial t^2} = c(\frac{\partial 2u}{\partial x^2} + \frac{\partial 2u}{\partial y^2}) \quad (2D\ wave\ equation)$$

$$6- \ \frac{\partial 2u}{\partial x^2} + \frac{\partial 2u}{\partial y^2} + \frac{\partial 2u}{\partial x^2} = 0 \quad (3D\ Laplace\ equation)$$

Example 4.1

The 2D Laplace equation is

$$\frac{\partial 2u}{\partial x^2} + \frac{\partial 2u}{\partial y^2} = 0$$

This equation has multiple solutions as

$$u(x, y) = x^2 + y^2$$

$$u(x, y) = e^x cosy$$

$$u(x, y) = \ln(x^2 + y^2)$$

All these solutions satisfy the equation and more solutions can be generated,

$$u(x, y) = C_1 u_1 + C_2 u_2 + C_3 u_3 + \cdots$$

This solution could be as

$$u(x, y) = \sum_{n=1}^{N} C_n u_n$$

Example 4.2

Solve

$$\frac{\partial 2E}{\partial x^2} = \frac{1}{C^2}\frac{\partial 2E}{\partial t^2}$$

Solution

Assume the general solution is $E(x,t) = R(x)K(t)$

$$\frac{\partial 2E}{\partial x^2} = K(t)\frac{\partial 2R(x)}{\partial x^2} \quad \textbf{and} \quad \frac{\partial 2E}{\partial t^2} = R(x)\frac{\partial 2K(x)}{\partial t^2}$$

$$K(t)\frac{\partial 2R(x)}{\partial x^2} = \frac{1}{C^2}R(x)\frac{\partial 2K(t)}{\partial t^2}$$

The goal now is to separate the two function R(x) and K(t)

By multiply both sides by $\frac{1}{R(x)K(t)}$ **the equation becomes**

$$\frac{1}{R(x)}\frac{\partial 2R(x)}{\partial x^2} = \frac{1}{C^2}\frac{1}{K(t)}\frac{\partial 2K(t)}{\partial t^2}$$

Now, we can vary (x) on the left side without changing the right side and at the same time we can change (t) on the left side without changing the right side,

That means both sides are constant

$$\frac{1}{R(x)}\frac{\partial 2R(x)}{\partial x^2} = (constant) \quad \textbf{and} \quad \frac{1}{C^2}\frac{1}{K(t)}\frac{\partial 2K(t)}{\partial t^2} = (constant)$$

Both equations above became Ordinary differential equations

They can be solved and the solution is

$$E(x,t) = R(x)K(t)$$

Example 4.3

Solve the following PDE $\frac{\partial 2u}{\partial x^2} = 4\,\frac{\partial 2E}{\partial y^2}$

Solution

Assume the general solution is

$$u(x,y) = R(x)K(y)$$

$$K(y)\frac{\partial 2R(x)}{\partial x^2} = 4R(x)\,\frac{\partial 2K(y)}{\partial y^2}$$

By separation

$$\frac{1}{4R(x)}\frac{\partial 2R(x)}{\partial x^2} = \frac{1}{K(y)}\,\frac{\partial K(x)}{\partial y} = -\lambda \quad \lambda\,(Separation\ constant)$$

$$\frac{1}{4R(x)}\frac{\partial 2R(x)}{\partial x^2} = -\lambda \qquad \frac{1}{K(y)}\,\frac{\partial K(x)}{\partial y} = -\lambda$$

$$R(x)^{//} + 4\lambda R(x) = 0 \quad \text{and} \quad K(y)^{/} + \lambda K(y) = 0$$

3 cases of λ

$\lambda = 0\ or\ < 0\ or > 0$

Case 1 $\lambda = 0$

$$\frac{\partial 2R(x)}{\partial x^2} = 0 \qquad \frac{\partial R(x)}{\partial x} = C_1 \qquad R(x) = C_1x + C_2$$

$$\frac{\partial K(y)}{\partial y} = 0 \quad K(y) = C_3$$

$$u(x,y) = R(x)K(y) = (C_1x + C_2)C_3$$

$$u(x,y) = A_1x + B_1$$

Case 2 $\lambda < 0$

$\lambda = -\alpha^2$

$$R(x)'' - 4\alpha^2 R(x) = 0 \quad and \quad K(y)' - \alpha^2 K(y) = 0$$

$$m = \pm 2\alpha \qquad\qquad m = \alpha^2$$

$$R(x) = C_4 e^{-2\alpha x} + C_5 e^{2\alpha x}, \qquad K(y) = C_6 e^{\alpha^2 y}$$

$$u(x,y) = R(x)K(y) = (C_4 e^{-2\alpha x} + C_5 e^{2\alpha x})\left(C_6 e^{\alpha^2 y}\right)$$

$$u(x,y) = A_2 e^{-2\alpha x + \alpha^2 y} + B_2 e^{2\alpha x + \alpha^2 y}$$

Case 3 $\lambda > 0$

$\lambda = \alpha^2$

$$R(x)'' + 4\alpha^2 R(x) = 0 \quad and \quad K(y)' + \alpha^2 K(y) = 0$$

$$m = \pm 2\alpha i \qquad\qquad m = -\alpha^2$$

$$R(x) = C_7 cos\, 2\alpha x + C_8 sin 2\alpha x, \qquad K(y) = C_9 e^{-\alpha^2 y}$$

$$u(x,y) = A_3 e^{-2\alpha y} \cos 2\alpha x + B_3 e^{-2\alpha y}\, sin 2\alpha y$$

Practice

1- Solve $\quad x\frac{\partial u}{\partial x} = t\frac{\partial u}{\partial t}$

2- Solve $\quad K\frac{\partial 2u}{\partial x^2} = \frac{\partial u}{\partial t}$

3- Show that $\quad u(x,t) = exp\left(-\frac{1}{\sqrt{4\pi t}}\frac{(x - x_o + 2t)^2}{4t}\right)$

Is a solution for $\quad \frac{\partial u}{\partial t} = \frac{\partial 2u}{\partial x^2} + 2\frac{\partial u}{\partial x}$ (Challenge)